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$z + \overline{z} = 2a$ is a real number
$z\overline{z} = a^2 + b^2$ is a non-negative real number

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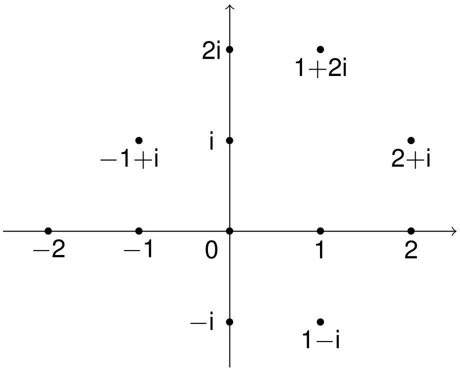
The x-axis is called the real axis.
The y-axis is called the imaginary axis.



Adding in the complex plane

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$$(a + b\mathrm{i}) + (c + d\mathrm{i}) = (a + c) + (b + d)\mathrm{i}.$$

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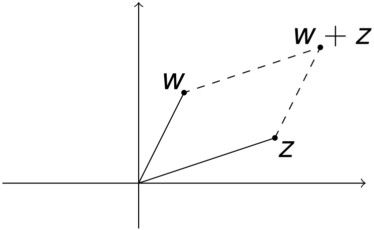
$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

In the complex plane, this is like adding vectors:



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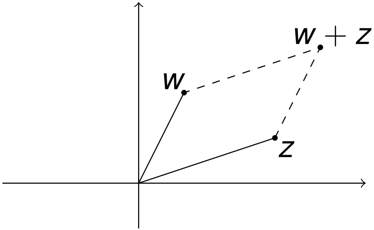

The points 0, z, w and $z + w$ form a parallelogram.

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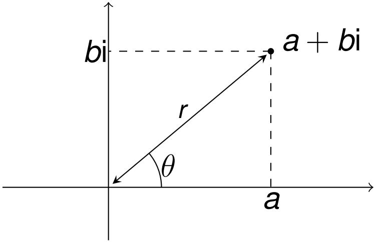

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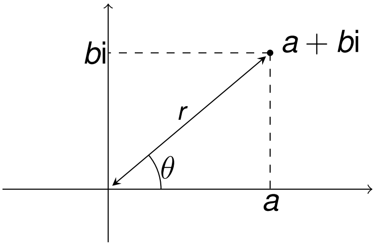

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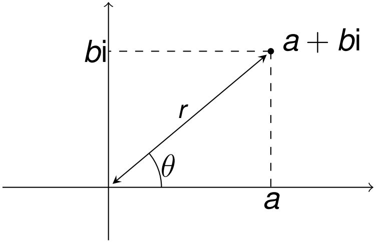

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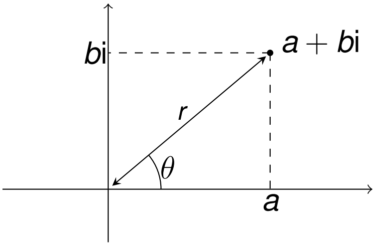

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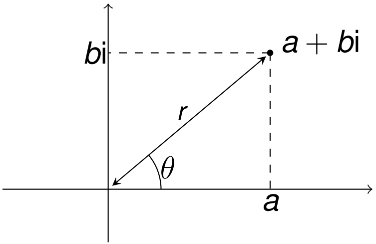

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r is the modulus of z, and θ is the argument of z.

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So multiplying by z means stretching by a factor $|z|$ and rotating through angle $\arg(z)$.

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(Proof by induction, using trig formulas.)

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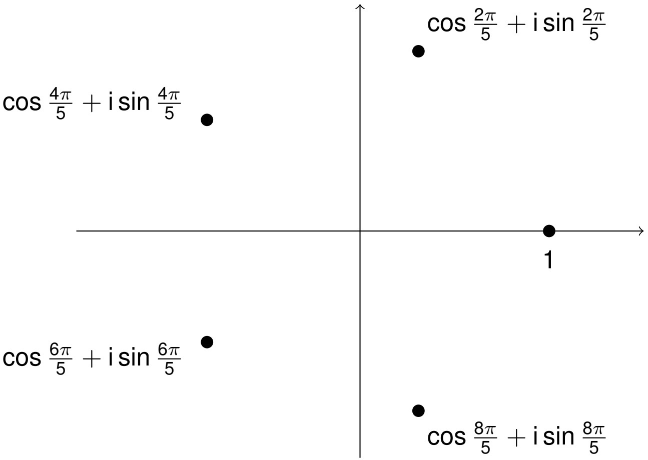
But in $\mathbb{C}$, De Moivre's Theorem shows that there are n solutions: these are the complex numbers

$$\cos\left(\frac{2\pi m}{n}\right) + \mathrm{i}\sin\left(\frac{2\pi m}{n}\right)$$

for $m \in \mathbb{Z}$. (In fact, just need $m = 0, 1, \ldots, n-1$.)

Solutions of $z^5 = 1$



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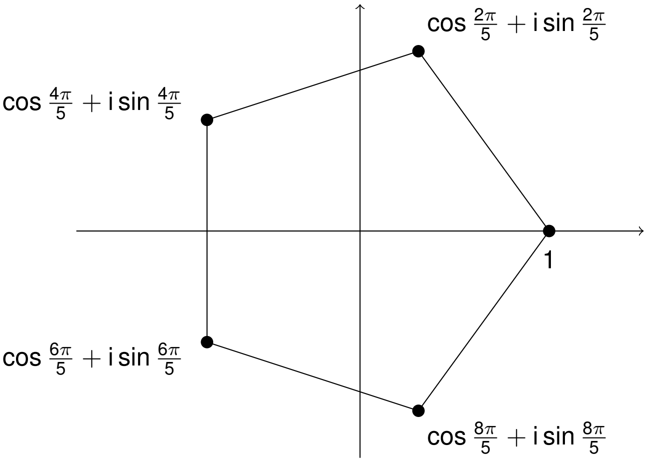

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In fact:

Fundamental Theorem of Algebra: Any equation of the form

$$z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0$$

(with $a_{n-1}, \ldots, a_1, a_0 \in \mathbb{C}$) has n roots in $\mathbb{C}$.