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Similarly

$$24.55299999\ldots = 24.553$$

etc.

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So extending from $\mathbb{Q}$ to $\mathbb{R}$ allows to to solve equations like $x^2 = 2$.

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if $X = (0, 1)$, then 1 is a supremum for X.

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Also write $\sup X = \infty$ if X is not bounded above, and $\sup \; = -\infty$.

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Then X is bounded above, but it has no maximum ...