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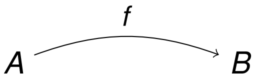

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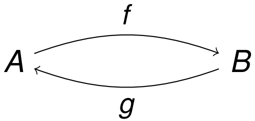

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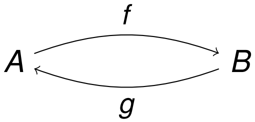

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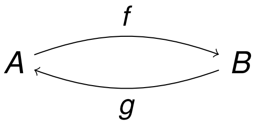

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We write the inverse of f as f^{-1}.

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But g is not an inverse for f, because $f(g(1)) \neq 1$.

f has no inverse: there is no $n \in \mathbb{N}$ such that $f(n) = 1$, so it's impossible to satisfy the condition $f(g(1)) = 1$.

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Suppose f is injective and surjective. Need to construct an inverse g.

Given $b \in B$, there is $a \in A$ such that $f(a) = b$ (because g is surjective).
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Then need to check that g is an inverse . . .

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For example, let $f : \{1, 2, 3, 4\} \to \{\text{red}, \text{green}, \text{blue}\}$ be defined by

$$1 \mapsto \text{green}, \qquad 2 \mapsto \text{red}, \qquad 3 \mapsto \text{red}, \qquad 4 \mapsto \text{blue}.$$

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(n.b. f is surjective but not injective, $f|_D$ is injective but not surjective.)

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We write $g \circ f$ for the composition of f and g.

$g \circ f$ means "do f then g".

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Define
$f : \mathbb{R} \to [0, \infty)$ by $f(x) = x^2$
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