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- choose the away team (19 choices).

So the total is $20 \times 19 = 380$.

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So $|\mathcal{P}(X)| = 2 \times 2 \times \cdots \times 2 = 2^n$.

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- $\binom{n}{0} = 1$: the only 0-element subset is $\emptyset$.
- $\binom{5}{2} = 10$: the 2-element subsets of $\{1, 2, 3, 4, 5\}$ are

$$\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\}.$$

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But now each k-element subset $\{a_1, a_2, \ldots, a_k\}$ has been counted $k!$ times (the number of ways of ordering $a_1, a_2, \ldots, a_k$), so divide by $k!$ to get the number of k-subsets.

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- $f : \mathbb{R} \to \mathbb{R}$ defined by

$$x \mapsto \begin{cases} 0 & (x < 0) \\ x & (x \geqslant 0). \end{cases}$$