Queen Mary
University of London

MTH6142: Complex Networks
Exam 2022 - Main Examination period
Marking scheme

Problem 1

a)

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \quad \boxed{3P}$$

The adjacency matrix is NOT symmetric. $\boxed{1P}$

b) The in-degree sequence is $\{0,1,1,1,3\}$ $\boxed{2P}$
The out-degree sequence is $\{3,1,1,1,0\}$ $\boxed{2P}$
The node with the largest in-degree is node 5, having $k_{max}^{in} = 3$. $\boxed{1P}$
The node with the largest out-degree is node 1, having $k_{max}^{out} = 3$. $\boxed{1P}$

c) $P_0^{in} = 1/5$, $P_1^{in} = 3/5$, $P_3^{in} = 1/5$, and $P_k^{in} = 0$ otherwise. $\boxed{2P}$
$P_0^{out} = 1/5$, $P_1^{out} = 3/5$, $P_3^{out} = 1/5$, $P_k^{out} = 0$ otherwise. $\boxed{2P}$

d) The Katz centrality x_i of a node i of a network is defined as: $\boxed{2P}$

$$x_i = \alpha \sum_{j}^{N} A_{ij} x_j + \beta$$

where $\alpha \in (0, 1/\lambda_1)$ and $\beta > 0$. Or in vectorial notation

$$\mathbf{x} = \beta \left(I - \alpha A\right)^{-1} \mathbf{1}$$

To find the Katz-centrality for all the nodes of the network G we need therefore to invert matrix $I - \alpha A$. In alternative, we can make use of the expression:

$$(I - \alpha A)^{-1} = \sum_{l=0}^{\infty} \alpha^l (A)^l \quad \boxed{2P}$$

We have:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \quad A^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \end{pmatrix} \quad A^l = 0 \quad l \geq 3$$

We thus have:

$$\begin{aligned} (I \quad - \quad \alpha A)^{-1} &= I + \alpha A + \alpha^2 A^2 = \\ &= I + \alpha \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} + \alpha^2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \end{pmatrix} = \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \alpha & 1 & 0 & 0 & 0 \\ \alpha & 0 & 1 & 0 & 0 \\ \alpha & 0 & 0 & 1 & 0 \\ 3\alpha^2 & \alpha & \alpha & \alpha & 1 \end{pmatrix} \quad \boxed{2P} \end{aligned}$$

In conclusion we have:

$$\mathbf{x} = \beta \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \alpha & 1 & 0 & 0 & 0 \\ \alpha & 0 & 1 & 0 & 0 \\ \alpha & 0 & 0 & 1 & 0 \\ 3\alpha^2 & \alpha & \alpha & \alpha & 1 \end{pmatrix} \mathbf{1}$$

and the centralities of the five nodes are: $\boxed{1P}$

$$\begin{aligned} x_1 &= \beta \\ x_2 = x_3 = x_4 &= \beta(1 + \alpha) \\ x_5 &= \beta(1 + 3\alpha + 3\alpha^2) \end{aligned}$$

e) There are no strongly connected components. $\boxed{1P}$

There is one weakly connected component $\boxed{1P}$ made by nodes (1,2,3,4,5) $\boxed{1P}$

f)

$$\mathbf{A}^u = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \quad \boxed{3P}$$

$L^u = 6$ links $\boxed{1P}$

g) The matrix of distances in the network is :

$$\mathbf{d} = \begin{pmatrix} 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 2 & 2 & 1 \\ 1 & 2 & 0 & 2 & 1 \\ 1 & 2 & 2 & 0 & 1 \\ 2 & 1 & 1 & 1 & 0 \end{pmatrix} \quad \boxed{\text{3P}}$$

The diameter D of G^u is $D = 2$ $\boxed{\text{1P}}$

h) The degree distribution $P(k)$ of network G^u is:

$P(0) = 0,\ P(1) = 0,\ P(2) = 3/5,\ P(3) = 2/5$ $\boxed{\text{2P}}$

i) The betweenness centrality of a node i of an undirected network is:

$$b_i = \sum_{r=1}^{N} \sum_{s=1}^{N} \frac{n^i_{rs}}{g_{rs}} \quad \boxed{\text{2P}}$$

where g_{rs} is the number of shortest paths between r and s, while n^i_{rs} is the number of shortest paths between r and s passing by i.

The betweenness centrality of node 1 is $b_1 = 9 + 6/2$ $\boxed{\text{2P}}$

The betweenness centrality of node 4 is $b_4 = 9 + 2/3$ $\boxed{\text{2P}}$

Problem 2

a) At time $t \geq 1$ the growth process of the model produces a graph with:

$$\begin{aligned} N(t) &= n_0 + (t-1) = 10 + t - 1 = 9 + t \qquad \text{nodes } \boxed{2\text{P}} \\ L(t) &= \binom{n_0}{2} + m(t-1) = 45 + 3(t-1) = 42 + 3t \qquad \text{links } \boxed{2\text{P}} \end{aligned}$$

b) The average node degree at time t is then:

$$\langle k \rangle = \frac{2L(t)}{N(t)} = \frac{2(42+3t)}{9+t} = \frac{84+6t}{9+t} \ \boxed{2\text{P}}$$

For $N \to \infty$, this corresponds to a network with an average degree

$$\langle k \rangle = 2m = 6 \ \boxed{2\text{P}}$$

c) The normalization sum is given by:

$$\begin{aligned} Z = \sum_{j=1}^{N(t-1)} k_j + \sum_{j=1}^{N(t-1)} a = 2L(t-1) + aN(t-1) = \\ = 2(42 + 3(t-1)) + a(9 + t - 1) = (6+a)t + 78 + 8a \end{aligned}$$

hence, for $t \gg 1$, $Z = \sum_j k_j - N(t) \simeq (6+a)t$ $\boxed{3\text{P}}$

d) In the mean-field approximation, the degree $k_i(t)$ of node i at time t satisfies the following differential equation

$$\frac{dk_i}{dt} = \tilde{\Pi}_i = 3\frac{k_i + a}{\sum_j (k_j + a)} \ \boxed{2\text{P}}$$

with initial condition $k_i(t_i) = 3$. Using the result found for the normalization sum $Z = (6+a)t$ in point c), for $t \gg 1$ we get:

$$\frac{dk_i}{dt} = 3\frac{k_i + a}{(6+a)t}.$$

Integrating the equation, with initial condition $k_i(t_i) = 3$:

$$\int_m^{k_i(t)} \frac{dk_i}{k_i + a} = \frac{3}{6+a} \int_{t_i}^{t} \frac{dt'}{t'} \ \boxed{2\text{P}}$$

we find the solution

$$k_i(t) = (3+a)\left(\frac{t}{t_i}\right)^{\frac{3}{6+a}} - a \ \boxed{3\text{P}}$$

e) The probability $P(k_i(t) > k)$ that a random node has degree $k_i(t) > k$, in the mean-field approximation can be calculated as follows

$$P(k_i(t) > k) = P\left[(3+a)\left(\frac{t}{t_i}\right)^{\frac{3}{6+a}} - a > k\right] = P\left[t_i < t\left(\frac{3+a}{k+a}\right)^{(6+a)/3}\right] \quad \boxed{2P}$$

Therefore we have

$$P(k_i(t) > k) = \left(\frac{3+a}{k+a}\right)^{(6+a)/3} \quad \boxed{1P}$$

Therefore, the degree distribution $P(k)$ is given by

$$P(k) = \frac{dP(k_i(t) < k)}{dk} = -\frac{d}{dk}\left(\frac{3+a}{k+a}\right)^{(6+a)/3} = C\left(\frac{1}{k+a}\right)^{(9+a)/3} \quad \boxed{3P}$$

with

$$C = \frac{6+a}{3}(3+a)^{(6+a)/3} \quad \boxed{1P}$$

f) The model produces indeed power-law degree distributions. In fact, for large k, we have

$$P(k) \propto k^{-\gamma} \quad \boxed{2P}$$

with an exponent $\gamma = 3 + a/3 > 3$ since $a > 0$. $\boxed{2P}$

g) The master equation for $N_k(t)$ reads

$$N_k(t+1) = N_k(t) + 3\frac{k-1+a}{(6+a)t}N_{k-1}(t) - 3\frac{k+a}{(6+a)t}N_k(t) \quad \text{for } k > 3 \quad \boxed{3P}$$

$$N_k(t+1) = N_k(t) - 3\frac{k+a}{(6+a)t}N_k(t) + 1 \quad \text{for } k = 3 \quad \boxed{3P}$$