Revision Lecture

Assessment

20% ASSESSED COURSEWORKS (5 Courseworks each accounting for 4%)

80% $\boxed{\text{FINAL EXAM}}$

ONLINE EXAM on QMPLUS

usually 2 or 3

Handwritten Questions for a total of 100 MARKS

Format (see "Assessment" on the webpage)

Under "Module Content"

9 MAY - 10AM

Exam papers

The final exam is a handwritten assessment that will be held online. The assessment will be **available for a period of 3.5 hours**, within which you must submit your solutions. You may log out and in again during that time, but the countdown timer will not stop. If your attempt is still in progress at the end of your 3.5 hours, any file you have uploaded will be automatically submitted.

The assessment has **been designed so that to be completed within 3 hours.** In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

For training and preparation, please see the Topic/Section: "Exam preparation and past papers", and consider the Formative Assignments of the module and the following past papers

2023 EXAM

Q 1 Network analysis

Q 2 Network modelling

2

Question 1 [50 marks].
Consider the undirected network $G = (V, E)$ with $N = 15$ nodes described by the following set of links:

$$E = \{(1,2), (1,3), (2,4), (2,5), (3,6), (3,7), (4,8), (4,9),$$

$$(5,10), (5,11), (6,12), (6,13), (7,14), (7,15)\}.$$

(a) Calculate the number of links L in the network, and draw the network. Is the network connected? Which type of network is this? (*Explain your answer*). **[7]**

(b) Write down the degree sequence, the degree distribution and the average degree of the network G. **[7]**

(c) Compute the diameter D of the network G, and list all the pairs of nodes which are at distance D (*Hint: it is not necessary to compute all the node distances to work out the diameter of the graph*). **[6]**

(d) Evaluate the betweenness centrality for all the nodes in the network G. Use the following definition of betweenness centrality of a node i:

$$b_i = \sum_{r=1}^{N} \sum_{s=1}^{N} \frac{n_{rs}^i}{n_{rs}}$$

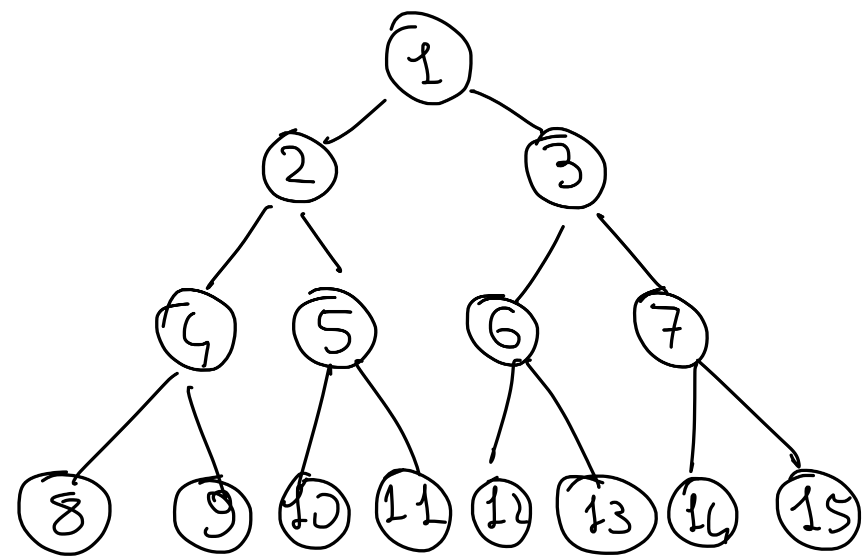
where n_{rs} is the number of shortest paths between node r and node s, while n_{rs}^i is the number of shortest paths between node r and node s passing through node i. *Hint: (Notice that, according to this definition, paths starting from node i or ending in node j are also counted).* **[11]**
i

(e) Which are the nodes with the largest possible values of betweenness centrality in the network, and which are those with the largest values of closeness centrality and of degree centrality? **[9]**

(f) Suppose you can add a link to connect two nodes of the network G to obtain another network G'. List one link choice that will create a triangle in the network G', one link choice that will create a cycle of length 4, and link choice that will create a cycle of length 5. How many are all the possible link choices that will guarantee that the network G' has a cycle of length $\ell = 7$. **[10]**

Q1

(a) The network has $L = 14$ links



The network is connected with $N = 15$ nodes and $L = 14 = N - 1$ links therefore it is a tree

(b) Degree sequence $\{2, \underbrace{3, 3}_{2}, \underbrace{3, 3, 3, 3}_{4}, \underbrace{1, 1, 1, 1, 1, 1, 1, 1}_{8}\}$

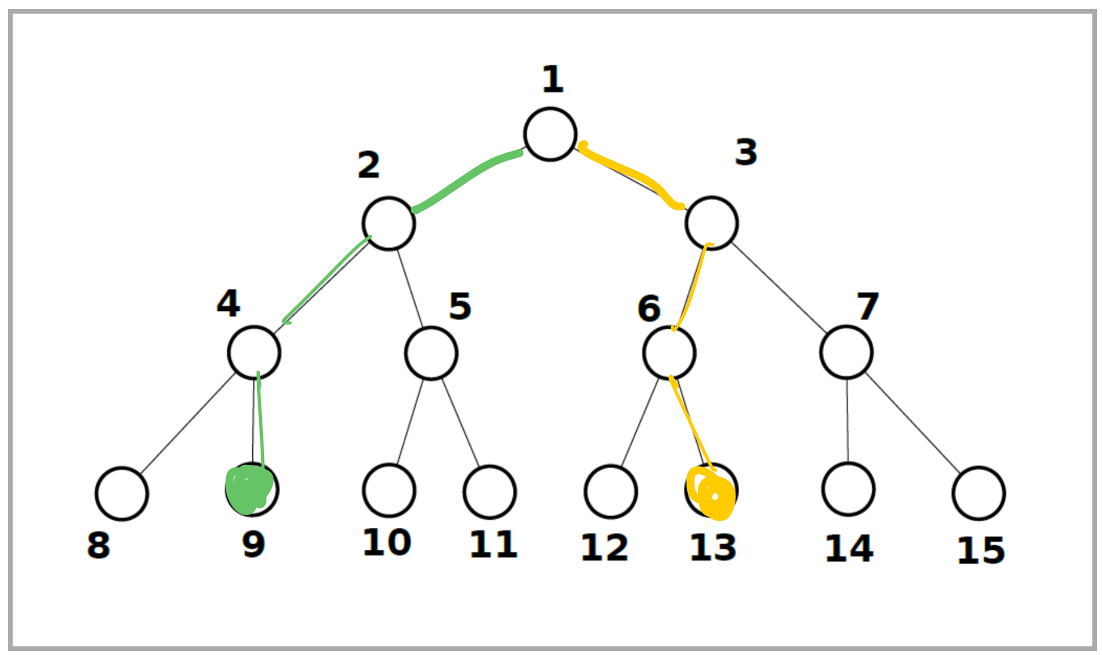
$$P_k = \begin{cases} 8/15 & \text{for } k = 1 \\ 1/15 & k = 2 \\ 6/15 & k = 3 \\ 0 & \text{otherwise} \end{cases}$$

$$\langle k \rangle = \frac{2L}{N} = \frac{28}{15}$$

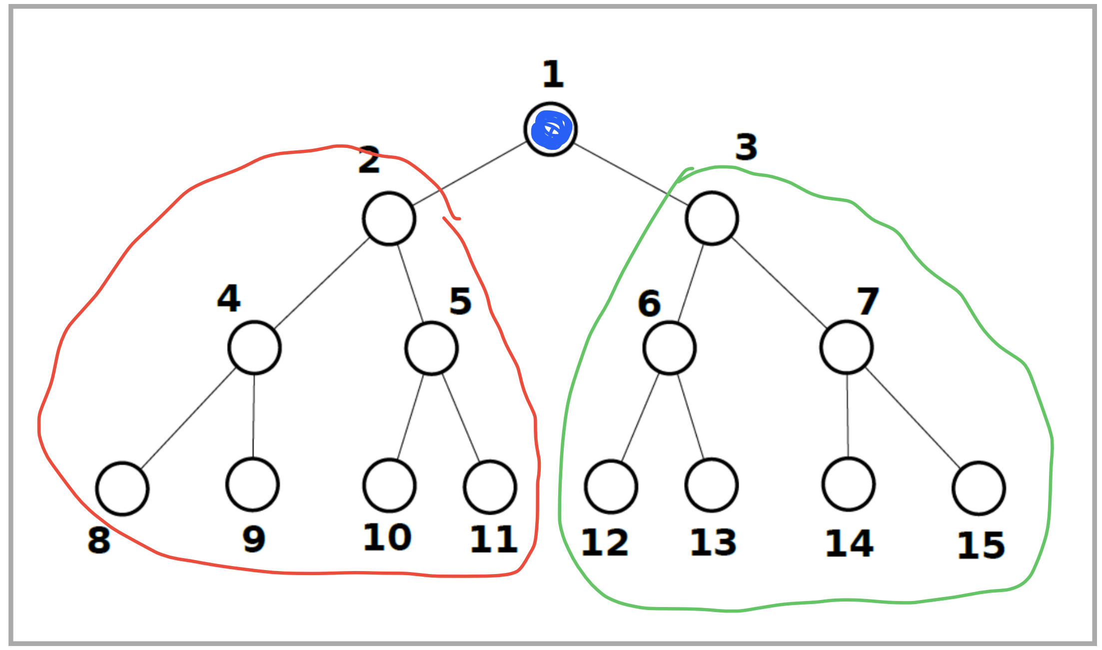
4

c) $D = 6$ (e.g distance between node 9 and 13)

Set of nodes at distance 6 is the Cartesian product of the two sets $V_1 = \{8, 9, 10, 11\}$ and $V_2 = \{12, 13, 14, 15\}$



d



$$b_i = \sum_{r=1}^{N} \sum_{s=1}^{N} \frac{n^i_{rs}}{\underbrace{n_{rs}}_{1}}$$

$n_{rs} = 1 \quad \forall r,s$ because the network is a tree

# of s.p. between r and s

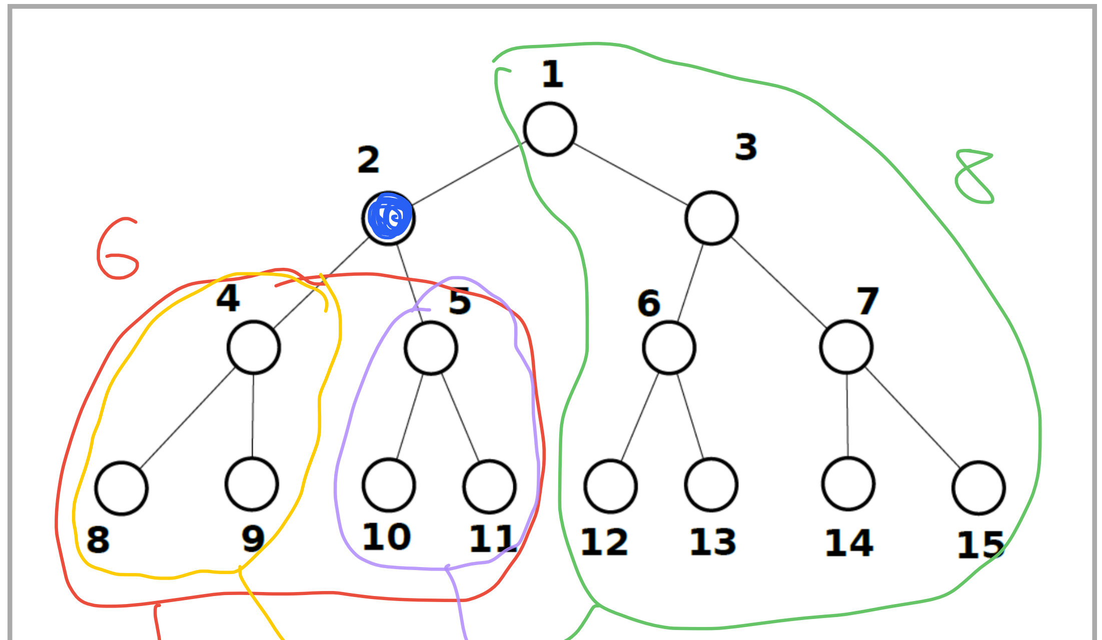
$$b_i = \sum_{r=1}^{N} \sum_{s=1}^{N} n^i_{rs}$$

paths starting or ending in i

$2N-1$

paths from red to green

$$C^B_1 = 2\cdot(7\cdot 7) + 2\cdot 15 - 1 = 2\cdot 49 + 29 = 127$$



$$C_2^B = C_3^B = 2\left[6 \cdot 8 + 3 \cdot 3\right] + 29 = 2 \cdot 57 + 29 = 143$$

$$C_4^B = C_5^B = C_6^B = C_7^B = 2\left[2 \cdot 12 + 1 \cdot 1\right] + 29 = 79$$

Nodes 8 - .... - 15 are all leaves with $C^B = 2N-1 = 29$

(e) largest betweenness : 2, 4

largest closeness : 1

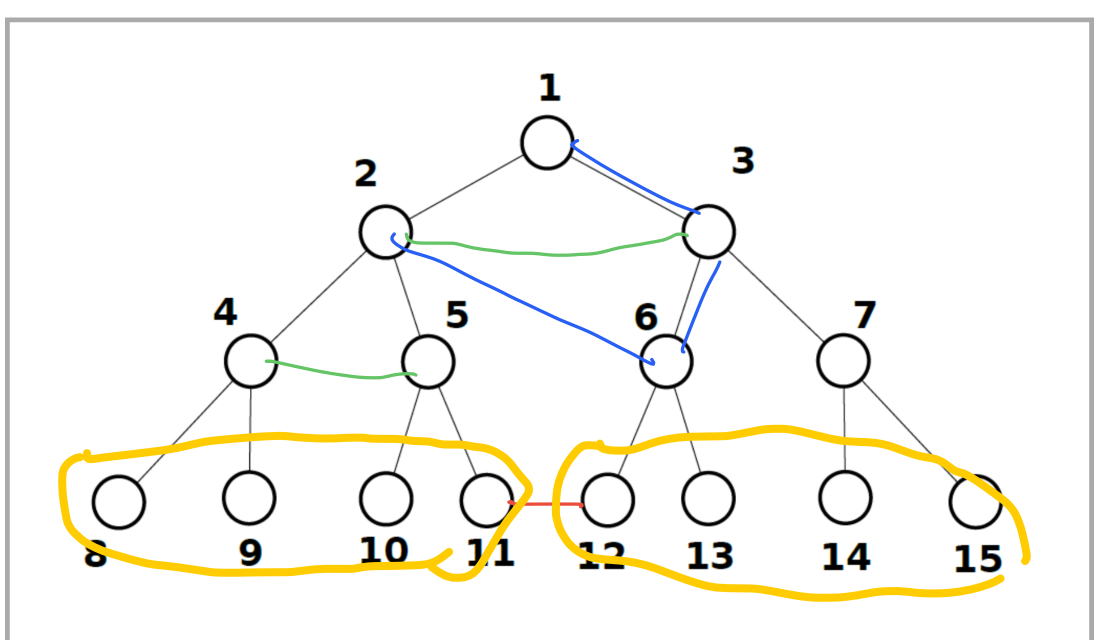
$$\sum_s d_{1s} = 2\cdot 1 + 4\cdot 2 + 8\cdot 3 = 34$$

$$\sum_s d_{2s} = 3\cdot 1 + 5\cdot 2 + 2\cdot 3 + 4\cdot 4 = 35$$

$34 < 35$

largest degree : 2, 3, 4, 5, 6, 7

(f)



(2, 3) → triangle

(2, 6) → cycle of length 4

(5, 6) → " length 5

(1, 12) → " length 7

16 possibilities

Question 2 [50 marks].

Consider the Erdös-Rènyi random graph ensemble $\mathbb{G}_{N,L}$ and assume that the number of links grows as a function of N, as $L(N) = \alpha N$, where α is a positive real constant (independent on N).

(a) Write down the general expression, as a function of N and α, for the probability of finding a given graph $G = (V, E)$ in the ensemble $\mathbb{G}_{N,L(N)}$. [**8**]

(b) Fix now $N = 5$ and consider the three cases $\alpha_1 = 0.8$, $\alpha_2 = 1$, and $\alpha_3 = 1.2$. Evaluate, in each of the three cases, the probability of finding a given graph $G = (V, E)$ with 5 nodes in the ensemble $\mathbb{G}_{N,L(N)}$. [**6**]

(c) Find a general expression for the average node degree $\langle k \rangle$ in a graph of the ensemble $\mathbb{G}_{N,L(N)}$, and evaluate the average node degree $\langle k \rangle_1$, $\langle k \rangle_2$ and $\langle k \rangle_3$ for the graphs in each of the three ensembles considered in point (b) (i.e. by fixing $N = 5$ and $\alpha_1 = 0.8$, $\alpha_2 = 1$, $\alpha_3 = 1.2$ respectively). [**7**]

(d) Find the function $p(N)$ of the Erdös-Rènyi random graph ensemble $\mathbb{G}_{N,p}$ which best approximate the Erdös-Rènyi random graph ensemble $\mathbb{G}_{N,L(N)}$. [**5**]

(e) For the Erdös-Rènyi random graph ensemble $\mathbb{G}_{N,p(N)}$ found in point (d), write down the distribution $P(L)$ for the total number of links L in a graph. Calculate the variance of such distribution and find an expression for the ratio r between the standard deviation and the expectation value of L. [**8**]

(f) Consider the Erdös-Rènyi random graph ensemble $\mathbb{G}_{N,p(N)}$ found in point (d), assume $N \to \infty$, and find the condition on α for the graphs to have a giant component. Find the same result using the Molloy-Reed criterion. [**8**]

(g) Consider the Erdös-Rènyi random graph ensemble $\mathbb{G}_{N,p(N)}$ found in point (d), assume $N \to \infty$, and find the expected numbers of triangles and 4-cliques as a function of α. [**8**]

a

$$G_{N,L} \leftarrow L = \alpha N$$

Prob of getting a graph $G=(V,E)$ $\rightarrow$

$$P(G) = \begin{cases} \frac{1}{Z} & \text{iff } |V| = N \text{ and } |E| = L \\ 0 & \text{otherwise} \end{cases}$$

$$Z = \binom{M}{L} = \frac{M!}{L!\,(M-L)!} \qquad M = \binom{N}{2} = \frac{N(N-1)}{2}$$

# of graphs with N nodes and L links

$L = \alpha N$

We get
$$P(G) = \begin{cases} \frac{1}{Z} & \text{if } |V| = N \text{ and } |E| = \alpha N \\ 0 & \text{otherwise} \end{cases}$$

where
$$Z = \binom{M}{\alpha N} = \frac{M!}{(\alpha N)!\,(M-\alpha N)!}$$

b

$$M = \binom{N}{2} = \binom{5}{2} = \frac{5 \cdot 4}{2} = 10$$ pairs of nodes

$$L_1 = \alpha_1 N = 0.8 \cdot 5 = 4$$

$$L_2 = \alpha_2 N = 1 \cdot 5 = 5$$

$$L_3 = \alpha_3 N = 1.2 \cdot 5 = 6$$

$$P(G) = \binom{10}{4}^{-1} = \frac{4!\,6!}{10!} = \frac{1}{210}$$ ← for a graph G of 5 nodes

$$P(G) = \binom{10}{5}^{-1} = \dots = \frac{1}{252}$$

$$P(G) = \binom{10}{6}^{-1} = \dots = \frac{1}{210}$$

(c)

$$\langle k \rangle = \frac{2 L(N)}{N} = \frac{2 \alpha \cancel{N}}{\cancel{N}} = 2\alpha$$

Hence $\langle k \rangle_1 = 1.6$ $\quad \langle k \rangle_2 = 2$ $\quad \langle k \rangle_3 = 2.4$

(d)

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– – See the PDF with the typed solutions