WEEK 12 Lecture 1

CHAPTER 8
THE CONFIGURATION MODEL

It is a model to generate random networks with a given degree sequence ← degree distribution

8.1 THE CONFIGURATION MODEL

DEF

Considers the ensemble of networks with N nodes and with a given **degree sequence** $\{K_i\} = \{K_1, K_2, ..., K_N\}$ constructed as follows:

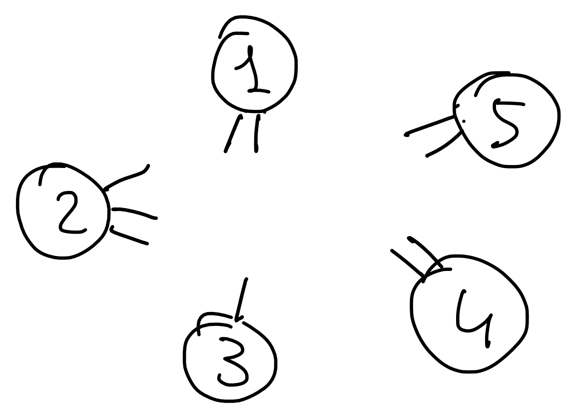
$\sum_i K_i = 2L$

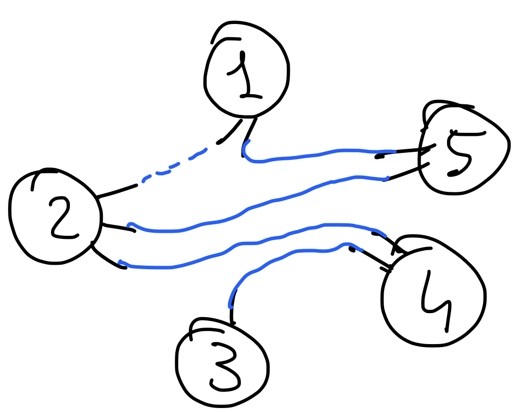
a) Place K_i stubs on node i

(b) Match each stub randomly with another stub

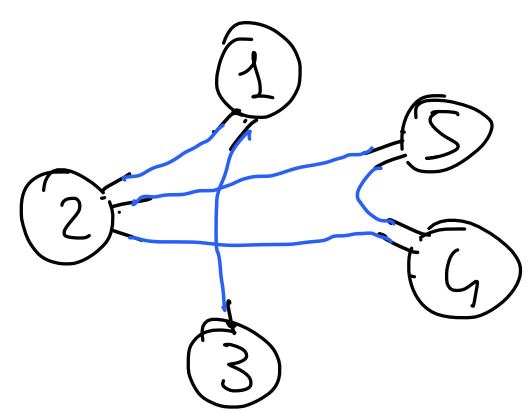
(c) When each stub is matched, if the network does not contain tadpoles or multiple links then stop, otherwise start again with (a)

Degree sequence $\{\overset{K_1}{2}, \overset{K_2}{3}, \overset{K_3}{1}, \overset{K_4}{2}, \overset{K_5}{2}\}$ $N = 5$ nodes

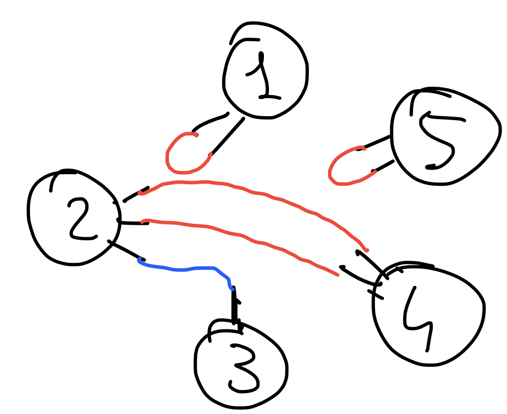




ACCEPT



ACCEPT



REJECT

8.2 UNCORRELATED NETWORKS

DEF

A network with degree sequence $\{k_i\}$ is UNCORRELATED if

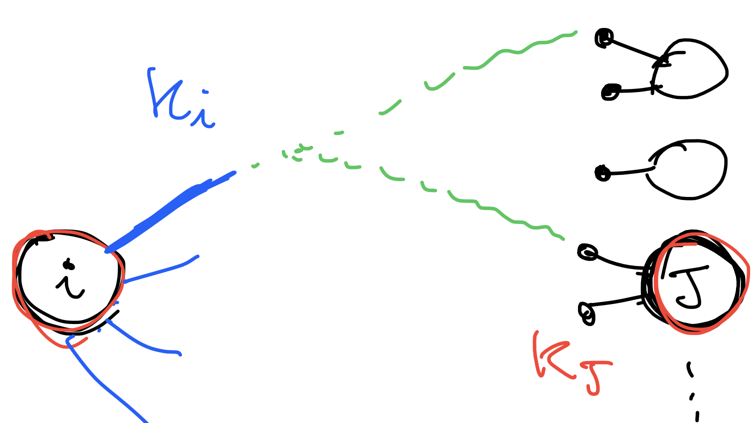
Probability that node i is connected to node j

$$\boxed{p_{ij} = \frac{k_i k_j}{\langle k \rangle N}}$$

Probability that a link from node i connects node j

$$\boxed{q_{ij} = \frac{k_j}{\langle k \rangle N} \equiv q_j}$$

uncorrelated if q_{ij} only depends on j



k_j possibilities to connect a link of i to node j, out of $\sum_j k_j = 2L = \langle k \rangle N$

$\langle k \rangle = \frac{2L}{N}$

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Hence $q_{ij} = q_j = \dfrac{k_j}{\langle k \rangle N}$ and $p_{ij} = k_i \cdot q_{ij} = \dfrac{k_i k_j}{\langle k \rangle N}$

DEF **STRUCTURAL CUTOFF**

The structural cutoff of a network is the maximum allowed degree given by

$$\boxed{K_s = \sqrt{\langle k \rangle N}}$$

A network has a structural cutoff if $k_i \le K_s \ \forall i$

PROPOSITION

The networks generated by the configuration model are UNCORRELATED iff they have a STRUCTURAL CUTOFF

Proof (of the necessary condition only)

a network is uncorrelated $\Rightarrow$ the network has the structural cutoff

$\Downarrow$

$$p_{ij} = \frac{k_i k_j}{\langle k \rangle N}$$

Let us indicate as $\bar{K}$ the largest degree in the network and assume

$k_i = k_j = \bar{K}$

Prob that i and j are connected $\rightarrow$ $P_{ij} = \frac{\bar{K} \cdot \bar{K}}{\langle k \rangle N}$

by definition $P_{ij} \leq 1$

$$\frac{\bar{K}^2}{\langle k \rangle N} \leq 1 \qquad \bar{K} \leq \sqrt{\langle k \rangle N}$$

$\bar{K}_s$

Since $\bar{K} = \max_i k_i$ we then have $k_i \leq \bar{K}_s \quad \forall i$

PROPOSITION

In an uncorrelated network the probability $q(k)$ that **following a link** we reach a node of degree k is

$$\boxed{q(k) = \frac{k \, P(k)}{\langle k \rangle}}$$

degree distribution

Proof

$$q_{ij} = q_j = \frac{k_j}{\langle k \rangle N}$$

Hence $\frac{k}{\langle k \rangle N}$ is the probability that following a link we get into a node j having $k_j = k$

probability that following a link we get in nodes of degree k

$$q(k) = \frac{k}{\langle k \rangle N} \cdot (\#\text{ of nodes of degree } k)$$

$$P(k) = \frac{N(k)}{N} \qquad N(k) = P(k) \cdot N$$

$$q(k) = \frac{k}{\langle k \rangle \cancel{N}} \cdot P(k) \cdot \cancel{N} = \frac{k\, P(k)}{\langle k \rangle}$$

The average node degree is $\langle k \rangle$, while the average degree of the neighbours of a node is NOT $\langle k \rangle$

PROPOSITION

The average degree k_{nn} (nearest-neighbours) **of the neighbours of a node** is given by

$$\boxed{k_{nn} = \frac{\langle k^2 \rangle}{\langle k \rangle}}$$

Proof

$$\langle k \rangle = \sum_k k\, p(k) \qquad \langle k^2 \rangle = \sum_k k^2\, p(k)$$

$$k_{nn} = \sum_k k\, q(k) = \sum_k k \frac{k\, p(k)}{\langle k \rangle} = \frac{1}{\langle k \rangle} \sum_k k^2 p(k) = \frac{\langle k^2 \rangle}{\langle k \rangle}$$

$$q(k) = \frac{k\, p(k)}{\langle k \rangle}$$

See example in FORM. ASSIGN 10

FRIENDSHIP PARADOX: your friends have more friends then you have!!

$\langle k \rangle$ = av # of friends of an individual

$k_{nn} = \frac{\langle k^2 \rangle}{\langle k \rangle}$ = av. # of friends of the neighbours of an individual

We have $k_{nn} > \langle k \rangle$ when $\frac{\langle k^2 \rangle}{\langle k \rangle} > \langle k \rangle$

$$\langle k^2 \rangle > \langle k \rangle^2$$

$$\langle (k - \langle k \rangle)^2 \rangle > 0$$

which is always true with the excention of regular networks where $k_i = \langle k \rangle \ \forall i$

8.4 CLUSTERING COEFFICIENT

PROPOSITION

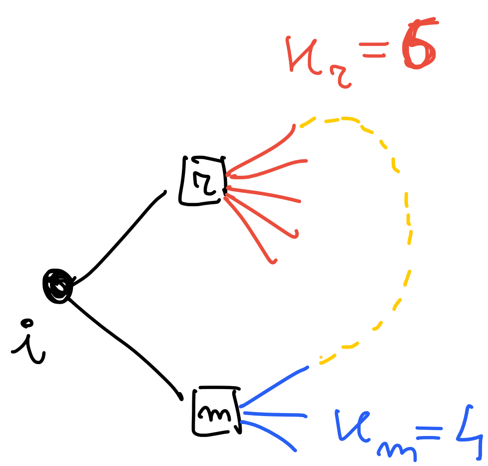
The clustering coefficient of an uncorrelated network from the configuration model is:

$$C = \frac{\langle k \rangle}{N}\left[\frac{\langle k^2\rangle - \langle k \rangle}{\langle k \rangle^2}\right]^2$$

Proof

$$C_i = \frac{T_i}{\frac{k_i(k_i-1)}{2}} = \text{fraction of pairs of neighbours of node } i \text{ that are linked}$$

probability that 2 neighbours of a node are linked



r with degree $k_r \to k_r - 1$ remaining links

m with degree $k_m \to k_m - 1$ remaining links

Since in the configuration model stubs are matched at random, then the probability that r is linked to m is

$$P_{rm} = \frac{(k_r - 1)(k_m - 1)}{\langle k \rangle N}$$

r has degree k_r with a probability $q(k_r)$

m " k_m " $q(k_m)$

Hence averaging over all possible values of k_r and k_m

$$C = \sum_{k_r} \sum_{k_m} q(k_r)\, q(k_m) \cdot \frac{(k_r - 1)(k_m - 1)}{\langle k \rangle N} =$$

$$= \frac{1}{\langle k \rangle N} \sum_{k_r} q(k_r)(k_r - 1) \cdot \sum_{k_m} q(k_m)(k_m - 1)$$

$$\sum_m \frac{k_m\, p(k_m)}{\langle k \rangle} (k_m - 1) =$$

$$= \frac{1}{\langle k \rangle N} \left[\frac{\langle k(k-1) \rangle}{\langle k \rangle} \right]^2 =$$

notice!!

$$= \frac{1}{\langle k \rangle} \sum_m k_m (k_m - 1)\, p(k_m)$$

$$= \frac{\langle k \rangle}{N} \left[\frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle^2} \right]^2$$

$$= \frac{\langle k(k-1) \rangle}{\langle k \rangle}$$

C^{Poiss}

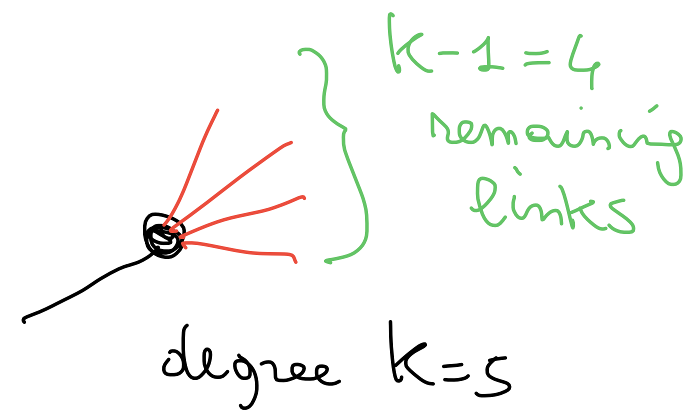
A correction term that can become large if $P(k)$ is scale-free which can lead to a finite C

8.5 AVERAGE DISTANCE

DEF **BRANCHING RATIO**

The branching ratio b_k of a node of degree k is

$$\boxed{b_k = k-1}$$



PROPOSITION

The average branching ratio $\bar{b}$ of a node reached by following a link in an uncorrelated network is:

$$\boxed{\bar{b} = \frac{\langle k(k-1)\rangle}{\langle k\rangle}}$$

used in Sect 7.6

Proof

$$\bar{b} = \sum_k \underbrace{b_k}_{k-1} \, g(k) = \sum_k (k-1)\, k \frac{p(k)}{\langle k\rangle} = \frac{1}{\langle k\rangle} \sum_k (k^2-k)\, p(k)$$

$$g(k) = \frac{k\,p(k)}{\langle k\rangle}$$

$$= \frac{1}{\langle k \rangle} \langle k(k-1) \rangle = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle}$$

PROPOSITION

The average distance ℓ in an uncorrelated network with finite average branching ratio $\bar{b}$ is, in the limit $N \to \infty$

$$\boxed{\ell \simeq \frac{\ln N}{\ln \frac{\langle k(k-1) \rangle}{\langle k \rangle}} + \text{constant}}$$

displays the S.W.D.P.

Proof

Assuming the network to be "locally tree-like" $\left(C \xrightarrow[N \to \infty]{} 0 \right)$

We compute the # of nodes at distance d from a given (generic) node

$$N_d \simeq \langle k \rangle \bar{b}^{\,d-1} = \langle k \rangle \left[\frac{\langle k(k-1) \rangle}{\langle k \rangle} \right]^{d-1}$$

average branching ratio $\bar{b} = \frac{\langle k(k-1) \rangle}{\langle k \rangle}$

We can now repeat the same argument as in Sect 7.6 where we had

$$N_d \simeq \langle k \rangle e^{d-1} \quad \text{and we found} \quad \ell \simeq \frac{\ln N}{\ln e}$$

and we get

$$\ell \simeq \frac{\ln \frac{N}{\langle k \rangle}}{\ln \frac{\langle k(k-1) \rangle}{\langle k \rangle}} + 1 \simeq \frac{\ln N}{\ln \frac{\langle k(k-1) \rangle}{\langle k \rangle}} + \text{Const.}$$

$$N \simeq \langle k \rangle \bar{b}^{\,\ell - 1}$$

$$\ln \frac{N}{\langle k \rangle} = (\ell - 1) \ln \bar{b}$$

$$\ell = \frac{\ln \frac{N}{\langle k \rangle}}{\ln \bar{b}} + 1$$