Queen Mary
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Complex Networks (MTH6142)
Solutions of Formative Assignment 9

- **1. Diameter of Cayley trees**
  A Cayley tree is a symmetric regular tree constructed starting from an origin node of degree k. In a Cayley tree every node at distance d from the origin node has degree k, until we reach the nodes at distance $\mathcal{P}$, which have degree equal to 1 and are called the leaves of the network.
  In this question take a Cayley tree with $k = 3$ as shown in the figure, and consider an arbitrary value of $\mathcal{P}$.

  a) Show using induction that the number of nodes at distance d from the central node is $3 \times 2^{d-1}$ for $d \in [1, \mathcal{P}]$.

  b) Using the formula for the sum of the first terms of a geometric series, show that the total number of nodes in the network is given by

  $$N = 1 + 3\left[2^{\mathcal{P}} - 1\right].$$

  c) Show that the diameter D is given by $D = 2\mathcal{P}$.

  d) Hence find an expression for the diameter D of the network in terms of the total number of nodes N.

  e) Consider the expression obtained in point (d) and find the leading term of D in terms of the total number of nodes N in the network when $N \gg 1$.

  f) Does the network display the small-world distance property?

  g) Which is the clustering coefficient of this network?

- *Notes on solution*

  a) Let us prove by iteration that the number of nodes N_d at distance d from the origin node is given by

  $$N_d = k(k-1)^{d-1}. \tag{1}$$

  The above relation is valid for the nodes at distance $d = 1$ from the origin node, since the degree of the origin node is k.
  Let us now show that if the relation (1) is valid for the nodes at distance d, then it must be valid for the nodes at distance $d + 1$.
  In fact we have that every node i at distance $d < \mathcal{P}$ from the origin

node has degree k, i.e. is linked to other k nodes.
Since the Cayley tree is connected and does not contain loops, only one of these k links is attached to a node at distance $d-1$ from the origin node while the other $k-1$ links are attached to nodes at distance $d+1$ from the origin node. It follows that each node at distance d will branch into $k-1$ nodes at distance $d+1$.
Moreover since the Cayley tree network does not contain loops any node at distance $d+1$ from the origin node can be reached only by one node at distance d.
Therefore we will have

$$N_{d+1} = N_d(k-1) = k(k-1)^{d-1}(k-1) = k(k-1)^d. \tag{2}$$

By putting $k=3$ we have

$$N_d = 3 \times 2^{d-1}. \tag{3}$$

b) The total number of nodes in the Cayley tree is given by the sum of 1 (indicating that there is only one origin node) at the sum of all the number of nodes N_d with distances $d \in [1, \mathcal{P}]$ from the origin node. Therefore we have

$$N = 1 + \sum_{d=1}^{\mathcal{P}} k(k-1)^{d-1} = 1 + k\frac{1-(k-1)^{\mathcal{P}}}{1-(k-1)}. \tag{4}$$

By putting $k=3$ we have

$$N = 1 + 3\left[2^{\mathcal{P}} - 1\right] \tag{5}$$

c) The maximal distance in the network is the distance between any two leaves nodes connected to the origin node by non-overlapping paths. Since the distance of any leaf node from the central node is $\mathcal{P}$, the diameter of the Cayley tree is given by $D = 2\mathcal{P}$.

d) Using $D = 2\mathcal{P}$ and using Eq. (5) we can derive the expression of D as a function of N, i.e.

$$\begin{aligned}
N &= 1 + 3\left[2^{D/2} - 1\right] \\
\left(\frac{N-1}{3}\right) &= 2^{D/2} - 1 \\
\left(\frac{N-1}{3}\right) + 1 &= 2^{D/2} \\
\frac{D}{2}\ln 2 &= \ln\left[1 + \frac{N-1}{3}\right] \\
D &= \frac{2}{\ln 2}\ln\left[1 + \frac{N-1}{3}\right].
\end{aligned} \tag{6}$$

e) Using Eq. (6) and considering the leading term for $N \gg 1$ we have

$$D \simeq 2\frac{\ln N}{\ln 2}. \tag{7}$$

f) The diameter of the network is $D \propto \ln N$ therefore the network displays the small-world distance property.

g) The network is a tree and has no triangles, therefore the clustering coefficient is zero for each node of the network and the Watts and Strogatz clusterign coefficient of the network is zero.

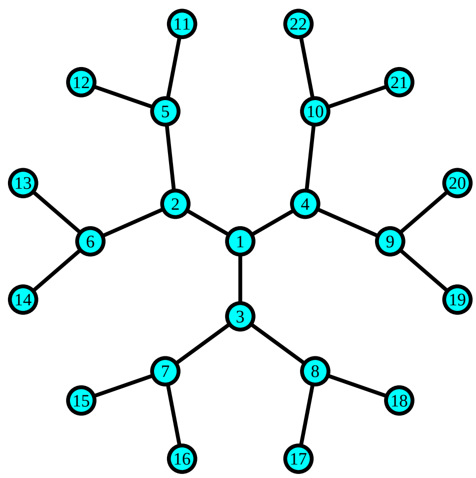

Figure 1: A Cayley tree network with $k = 3$ and $\mathcal{P} = 3$.