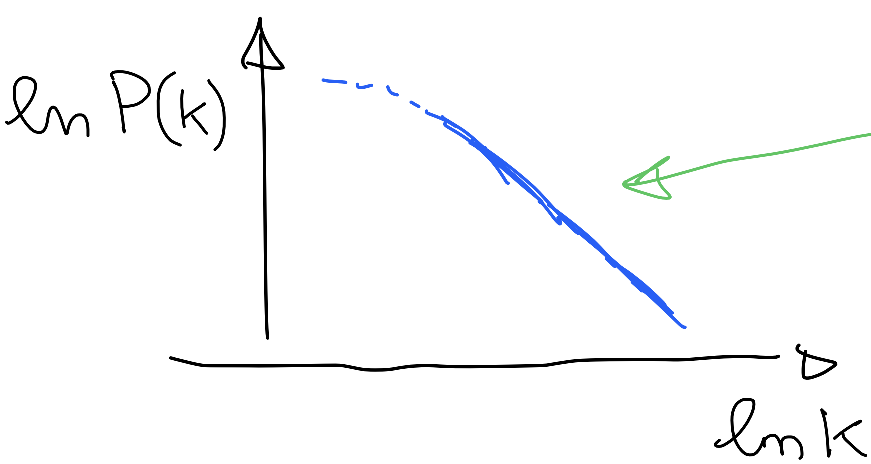
WEEK 11 Tutorial

TAKE-HOME MESSAGE from W 8-9-10

SCALE-FREE NETWORKS

$$P(k) \simeq C\, k^{-\gamma} \qquad k \gg 1$$



$\gamma \in (2, 3]$

$\langle k \rangle$ is finite

$\langle k^2 \rangle$ diverges

PROPERTIES of POWER-LAWS (continuous approximation)

$$P(k) = C\, k^{-\gamma} \qquad k \in \mathbb{R} \qquad k \in [k_{min}, K]$$

$$C \simeq (\gamma - 1)\, k_{min}^{\gamma - 1}$$

$$\bar{K} \simeq \begin{cases} K_{min} \, N^{\frac{1}{\gamma-1}} & \text{for } \gamma > 2 \\ N & \text{for } \gamma \, (1,2] \end{cases}$$

$$\langle k^n \rangle = \begin{cases} \dfrac{C}{n+1-\gamma} \left[\bar{K}^{\,n+1-\gamma} - k_{min}^{\,n+1-\gamma} \right] & \text{for } n \neq \gamma - 1 \\ C \, \ln \left[\dfrac{\bar{K}}{k_{min}} \right] & \text{for } n = \gamma - 1 \end{cases}$$

$\langle k \rangle \longrightarrow \infty$ iff $\gamma \leq 2$

$\langle k^2 \rangle \longrightarrow \infty$ iff $\gamma \leq 3$

MODELS OF NETWORK GROWTH

	BA MODEL	UNIFORM ATTACHMENT	BB MODEL
$\Pi_i =$	$\dfrac{k_i}{\sum_j k_j}$	$\dfrac{1}{N(t)}$	$\dfrac{\eta_i \, k_i}{\sum_j \eta_j \, k_j}$

η from $\rho(\eta)$

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MF equations

$$\begin{cases} \dfrac{dk_i(t)}{dt} = \tilde{\Pi}_i = m\,\Pi_i & \text{for } t > t_i \\ k_i(t_i) = m \end{cases}$$

$k_i(t) =$ $t \geq t_i$ time when node i was introduced	$m\left(\dfrac{t}{t_i}\right)^{\frac{1}{2}}$	$m + m\ln\left(\dfrac{t}{t_i}\right)$	$m\left(\dfrac{t}{t_i}\right)^{\frac{\eta_i}{C}}$ $\displaystyle\int \frac{\rho(\eta)}{\frac{C}{\eta} - 1}\, d\eta = 1$
$P(k) \simeq$	$2m^2 k^{-3}$ $\gamma = 3$	$\dfrac{e}{m}\, e^{-\frac{k}{m}}$	$\displaystyle\int d\eta\, \rho(\eta)\, \frac{C}{\eta}\, \frac{m^{\frac{C}{\eta}}}{k^{1+\frac{C}{\eta}}}$

$$\mathrm{Prob}(t_i < \tau) \simeq \frac{\tau}{t} \to \mathrm{Prob}(k_i(t) > k) \to P(k) = \frac{d\,\mathrm{Prob}(k_i(t) \leq k)}{dk}$$

EXACT

$P(k)$

$$\frac{2m(m+1)}{k(k+1)(k+2)}$$

$$\frac{e^{m \ln\left(1+\frac{1}{m}\right)}}{1+m} e^{-k \ln\left(1+\frac{1}{m}\right)}$$

Costant

MASTER EQUATION

GAIN

LOSS

$$N_k(t+1) = N_k(t) + \tilde{\Pi}(k-1) N_{k-1}(t) - \tilde{\Pi}(k) N_k(t)$$

for $k > m$

$$N_k(t+1) = N_k(t) + 1 - \tilde{\Pi}(k) N_k(t)$$

for $k = m$

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From FA 7

At time $t = 0$ the network is formed by a $n_0 = 2$ nodes and a single link (initial number of links $m_0 = 1$) connecting the two nodes.
At every time step $t > 0$ the network evolve according to the following rules:

- A single new node joins the network.
- A link (i, r) between a node i and a node r is chosen randomly with uniform probability

$$\pi_{(i,r)} = \frac{A_{i,r}}{L}$$

and the new node is linked to both node i and node r.

a) Show that in this network evolution at each time step the average number of links $\tilde{\Pi}_i$ added to node i follows the preferential attachment rule, i.e.

$$\tilde{\Pi}_i = \sum_{r=1}^{N} \pi_{(i,r)} = 2\frac{k_i}{\sum_{j=1}^{N} k_j}.$$

b) What is the total number of links in the network at time t? What is the total number of nodes?

c) What is the average degree $\langle k \rangle$ of the network at time t?

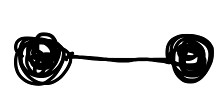
d) Use the result at point a) to derive the time evolution $k_i = k_i(t)$ of the average degree k_i of a node i for $t \gg 1$ in the mean-field, continuous approximation.

e) What is the degree distribution of the network at large times in the mean-field approximation?

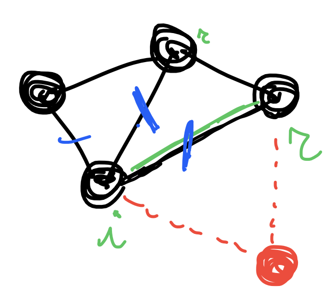
$t = 0$

$m_0 = 2$

$m_0 = 1$



$t > 0$



$m = 2$

link (i, r) is chosen with

$$\Pi(i,r) = \frac{A_{ir}}{L}$$

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At time t we have $N(t)$ nodes and $L(t)$ links

(a)

$$\tilde{\Pi}_i = \sum_{r=1}^{N} \Pi(i,r) = \sum_{r=1}^{N} \frac{A_{ir}}{L} = \frac{k_i}{\frac{1}{2}\sum_{j} k_j} = \frac{2k_i}{\sum_j k_j}$$

$$L = \frac{1}{2}\sum_{j=1}^{N} k_j$$

$$k_i = \sum_{r=1}^{N} A_{ir}$$

(b)

$$L(t) = m_0 + m\,t = 1 + 2t$$

$m_0 = 1$, $m = 2$

$$N(t) = m_0 + t = 2 + t$$

$m_0 = 2$

(c)

$$\langle k \rangle = \frac{2L}{N} = \frac{2(1+2t)}{2+t}$$

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d

MF $\quad \dfrac{d k_i(t)}{dt} = \tilde{\Pi}_i = \dfrac{2 k_i}{\sum_j k_j}$

$\sum_j k_j = 2L = 2(1+2t) \simeq 4t$

$t \gg 1$

$$\begin{cases} \dfrac{d k_i}{dt} = \dfrac{\cancel{2}^{1} k_i}{\cancel{4}^{2} t} = \dfrac{k_i}{2t} & t > t_i \\ k_i(t) = m = 2 & t = t_i \end{cases}$$

Integrating

$$\int_{m}^{k_i(t)} \frac{d k_i'}{k_i'} = \frac{1}{2} \int_{t_i}^{t} \frac{dt'}{t'}$$

m 2

$$\ln \frac{k_i(t)}{m} = \frac{1}{2} \ln \frac{t}{t_i}$$

m 2

$$k_i(t) = m \left(\frac{t}{t_i}\right)^{\frac{1}{2}} =$$

$$= 2 \left(\frac{t}{t_i}\right)^{\frac{1}{2}}$$

$$k_i(t) = 2 \left(\frac{t}{t_i}\right)^{\frac{1}{2}}$$

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(e)

$$\mathrm{Prob}\left(k_i(t) > k\right) = \mathrm{Prob}\left(2\left(\frac{t}{t_i}\right)^{\frac{1}{2}} > k\right) =$$

$$= \mathrm{Prob}\left(t_i < \underbrace{\left(\frac{2}{k}\right)^2 t}_{\tau}\right) \simeq \frac{\left(\frac{2}{k}\right)^2 \cancel{t}}{\cancel{t}} = \left(\frac{2}{k}\right)^2$$

$$\mathrm{Prob}(t_i < \tau) \simeq \frac{\tau}{t}$$

$$P(k) = \frac{d\,\mathrm{Prob}\left(k_i(t) \leq k\right)}{dk} = -\frac{d}{dk}\left(\frac{2}{k}\right)^2 = -2^2 \frac{d}{dk} k^{-2} =$$

$$= 2^3 k^{-3}$$

$\gamma = 3$

ASSESSED ASSIGNMENT 5 CLOSES TODAY