Queen Mary
University of London

Complex Networks (MTH6142)
Solutions of Formative Assignment 8

- **1. Growing network model with uniform attachment**
  At time $t = 0$ the network is formed by two nodes joined by a link.

  – At every time step a single new node joins the network, so that at time t there will be exactly $N(t) = 2 + t$ nodes.
  Every new node has initially $m = 1$ links.

  – Each new link attaches to an existing node of the network. The target node i is chosen with probability Π_i following a uniform attachment rule $\Pi_i = \frac{1}{N(t)}$, where k_i is the degree of the node i.

  a) What is the time evolution of the degree $k_i(t)$ of the generic node i in the mean-field approximation?
  b) What is the degree distribution of the network at large times in the mean-field approximation?
  c) What is the master equation for the average number of nodes with degree k?
  d) Solve the master equation finding the exact degree distribution of the network.

- *Notes on solution* a) In the mean-field approximation, the average degree of a node i, $k_i(t)$ follows the following differential equation

$$\frac{dk_i}{dt} = \frac{m}{N(t)} \tag{1}$$

with initial condition $k_i(t_i) = m$ where t_i is the time of arrival of the node i in the network. Here $N(t) = n_0 + t$ indicates the total number of nodes in the network. In the large network limit $t \gg 1$ we can approximate $N(t)$ as $N(t) = n_o + t \simeq t$. Integrating the differential equation for $k_i(t)$ we get

$$k_i(t) = m \ln\left(\frac{t}{t_i}\right) + m = m \ln\left(e\frac{t}{t_i}\right). \tag{2}$$

b) In the mean-field equation we can calculate the probability $P(k_i(t) > k)$ that a random node of the network has degree greater than k. This probability is given by

$$\begin{aligned} P(k_i(t) > k) &= P\left(m \ln\left(e\frac{t}{t_i}\right) > k\right) \\ &= P\left(t_i < te^{1-k/m}\right) \\ &= e^{1-k/m}, \end{aligned} \tag{3}$$

in fact $P(t_i < \tau) = \tau/t$. The degree distribution $P(k)$ in the mean-field approximation is given by

$$P(k) = \frac{d[1 - P(k_i > k)]}{dk} = \frac{1}{m}e^{1-k/m} \tag{4}$$

Therefore the degree distribution is exponential in the mean-field approximation.
c)The master equation for the average number $N_k(t)$ of nodes of degree k at time t is given by

$$\begin{array}{ll} N_k(t+1) = N_k(t) + m\Pi(k-1)N_{k-1}(t) - m\Pi(k)N_k(t) & \text{for} \quad k > m \\ N_k(t+1) = N_K(t) + 1 - m\Pi(k)N_k(t) & \text{for} \quad k = m. \end{array}$$

with $\Pi(k) = \frac{1}{N(t)} = \frac{1}{n_0+t}$.
d) For $t \gg 1$ $\Pi(k) \simeq \frac{1}{t}$ and $N_k(t) \simeq tP(k)$ yielding

$$P(k) = mP(k-1) - mP(k) \tag{5}$$

for $k > m$ and

$$P(m) = -mP(m) + 1 \tag{6}$$

for $k = m$. Therefore we have

$$P(k) = \frac{m}{1+m}P(k-1) \tag{7}$$

for $k > m$ and

$$P(m) = \frac{1}{1+m}. \tag{8}$$

Using these equations recursively we get the explicit expression for the degree distribution

$$P(k) = \left(\frac{m}{1+m}\right)^{k-m} \frac{1}{1+m}. \tag{9}$$

This is the exact result for the degree distribution of the model. This is confirming that the degree distribution is an exponential for the uniformly growing network model.

- **2. The Bianconi-Barabási model with uniform fitness distribution**
  Each node i of the network has a fitness value η_i drawn from a uniform distribution $\rho(\eta) = 1$ for $\eta \in [0, 1]$.
  At time $t = 0$ the network is formed by two nodes joined by a link.
  - At every time step a single new node j with fitness η_j drawn from the $\rho(\eta)$ distribution joins the network, so that at time t there will be exactly $N = 2+t$ nodes.Every new node has initially $m = 1$ links.
  - Each new link attaches to an existing node of the network. The target node i is chosen with probability Π_i following the preferential attachment rule biased toward nodes of high degree and high fitness $\Pi_i = \frac{\eta_i k_i}{\sum_j \eta_j k_j}$, where k_i is the degree of the node i.

  a) Assume that $\sum_j \eta_j k_j \simeq Ct$ for $t \gg 1$, with $C > 0$ indepedennt on the time t. What is the time evolution of the degree $k_i(t)$ of the generic node i in the mean-field approximation?
  b) Check self-consistently that the assumption made in part a) is correct and determine the equation that the constant C needs to satisfy.

- *Notes on solution*
  a)In the mean-field approximation, the degree $k_i(t)$ of node i at time t satisfies the following differential equation

$$\frac{dk_i(t)}{dt} = \frac{\eta_i k_i}{\sum_j \eta_j k_j} \tag{10}$$

We assume self-consistently that the normalization sum $\sum_j \eta_j k_j$ has the limiting behaviour $\sum_j \eta_j k_j \simeq Ct$ for $t \gg 1$, therefore

$$\lim_{t\to\infty} \frac{\sum_j \eta_j k_j}{t} = C \tag{11}$$

with $C > 0$. In this hypothesis, and in the limit $t \gg 1$ we can write the dynamical mean-field equation for the degree $k_i(t)$ of node i, getting

$$\frac{dk_i}{dt} = \frac{\eta_i k_i}{Ct}, \tag{12}$$

with initial condition $k_i(t_i) = 1$. This equation has solution

$$k_i(t) = \left(\frac{t}{t_i}\right)^{\eta_i/C}. \tag{13}$$

The us note that, since the total number of links is increasing linearly with time, the degree of the nodes in the network cannot grow faster than

linearly in time. Therefore let us assume $\eta/C < 1$.
b)In order to check if our assumption in Eq. (11) is consistent with the solution (13) we note that Eq. (11) implies

$$\lim_{t\to\infty} \frac{\left\langle \sum_j \eta_j k_j \right\rangle}{t} = C \tag{14}$$

Now the quantity $\left\langle \sum_j \eta_j k_j \right\rangle$ can be calculated using the solution Eq. (**??**) and the continuous approximation , getting the self consistent equation

$$\int d\eta\rho(\eta) \int_1^t dt_j \eta \left(\frac{t}{t_j}\right)^{\eta/C} = Ct \tag{15}$$

for $t \gg 1$. Performing the integral we get

$$\begin{aligned} &\int d\eta\rho(\eta) \int_1^t dt_j \eta \left(\frac{t}{t_j}\right)^{\eta/C} \\ &\int d\eta\rho(\eta) \frac{\eta}{1-\eta/C} \left[t - t^{\eta/C}\right] \end{aligned} \tag{16}$$

Since $\eta/C < 1$, if we perform the limit in Eq. (14), we get a self-consistent equation for the constant C given by

$$C = \int d\eta\rho(\eta) \frac{\eta}{1-\eta/C} \tag{17}$$

this equation can be also written as

$$1 = \int d\eta\rho(\eta) \frac{1}{C/\eta - 1}. \tag{18}$$