WEEK 10 Lecture 2

fitness of node i

from $\rho(\eta)$

BB model

$$\Pi_i = \frac{\eta_i k_i}{\sum_j \eta_j k_j}$$

- In lecture 1 we derived (in the MF approximation)

$$K_i(t) = m\left(\frac{t}{t_i}\right)^{\frac{\eta_i}{C}} \quad \text{for } t \geq t_i$$

$$\int \frac{\rho(\eta)}{\frac{C}{\eta} - 1}\, d\eta = 1$$

- We will now derive $P(k)$

PROPOSITION

In the **MF** approximation the degree distribution of the **BB** model is given in the limit $t \to \infty$ by

$$\boxed{P(k) = \int d\eta \, \rho(\eta) \, \frac{C}{\eta} \, m^{\frac{C}{\eta}} \, \frac{1}{k^{1+\frac{C}{\eta}}}}$$

Moreover it can be proven that this corresponds to a power-law $P(k)$ with an exponent in the range $\gamma \in (2,3]$ and with a "logarithmic correction"

$$P(k) \simeq \frac{k^{-(1+C)}}{\ln k}$$

Proof We first evaluate

$$\underbrace{\text{Prob}\left(k_i(t) > k \mid \eta_i = \eta\right)}_{\substack{\text{Probability that a node } i \\ \text{with fitness } \eta \text{ has} \\ k_i(t) > k}} = \text{Prob}\left(m\left(\frac{t}{t_i}\right)^{\frac{\eta}{C}} > k\right) =$$

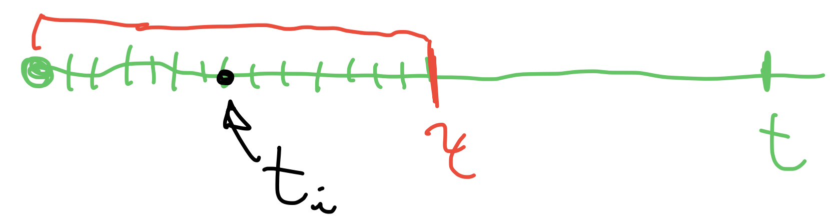
Using

$$k_i(t) = m\left(\frac{t}{t_i}\right)^{\frac{\eta_i}{C}}$$

$$= \text{Prob}\left(\frac{t}{t_i} > \left(\frac{k}{m}\right)^{\frac{c}{\eta}}\right) = \text{Prob}\left(t_i < \underbrace{t\left(\frac{m}{k}\right)^{\frac{c}{\eta}}}_{\tau}\right) \simeq$$

Using

$$\text{Prob}(t_i < \tau) \simeq \frac{\tau}{t}$$



$$\simeq \frac{\cancel{t}\left(\frac{m}{k}\right)^{\frac{c}{\eta}}}{\cancel{t}} = \left(\frac{m}{k}\right)^{\frac{c}{\eta}}$$

Finally

$$P(k|\eta) = \frac{d\,\text{Prob}\left(k_i(t) \le k \,\middle|\, \eta_i = \eta\right)}{dk} =$$

degree distribution of nodes with fitness η

$$= \frac{d}{dk}\left[1 - \text{Prob}\left(k_i(t) > k \,\middle|\, \eta_i = \eta\right)\right] =$$

$$\text{Prob}\left(k_i(t) > k \,\middle|\, \eta_i = \eta\right) = \left(\frac{m}{k}\right)^{\frac{c}{\eta}}$$

$$= -\frac{d}{dk}\left(\frac{m}{k}\right)^{\frac{c}{\eta}} = -\frac{c}{\eta}\left(\frac{m}{k}\right)^{\frac{c}{\eta}-1}\left(-\frac{m}{k^2}\right) =$$

$$= \frac{c}{\eta} \, m^{\frac{c}{\eta}} \, \frac{1}{k^{\frac{c}{\eta}+1}}$$

Now we can obtain $P(k)$

$$P(k) = \int P(k|\rho) \, \rho(\eta) \, d\eta = \int d\eta \, \rho(\eta) \, \frac{c}{\eta} \, m^{\frac{c}{\eta}} \, \frac{1}{k^{\frac{c}{\eta}+1}}$$

which is the

Let us now consider some special cases of $\rho(\eta)$

$\boxed{\text{LIMIT CASE 1}}$ All the nodes with the SAME FITNESS

We should recover the BA model

$\eta_i = 1 \quad \forall i$

$\downarrow$ in terms of $\rho(\eta)$

$\rho(\eta) = \delta(\eta - 1)$ where $\delta(x - x_0)$ is the DIRAC DELTA function for which $\int dx \, \delta(x - x_0) \cdot f(x) = f(x_0)$

$$\int d\eta \, \delta(\eta - 1) f(\eta) = f(1)$$

- Eq ✿✿ for $\eta_i = 1$ $\forall i$ becomes $k_i(t) = m\left(\frac{t}{t_i}\right)^{\frac{1}{C}}$

with C satisfying

$$1 = \int d\eta \, \frac{\rho(\eta)}{\frac{C}{\eta} - 1} = \int d\eta \, \delta(\eta - 1) \overbrace{\frac{1}{\frac{C}{\eta} - 1}}^{f(\eta)} = \frac{1}{\frac{C}{1} - 1} = \frac{1}{C-1}$$

$$1 = \frac{1}{C-1} \qquad C = 2$$

Hence $k_i(t) = m\left(\frac{t}{t_i}\right)^{\frac{1}{C}} = m\left(\frac{t}{t_i}\right)^{\frac{1}{2}}$

which coincides with $\circledast \circledast$ of BA model

- Eq ✿✿✿

$C = 2$, $\rho(\eta) = \delta(\eta - 1)$

$$P(k) = \int d\eta \, \rho(\eta) \, \frac{C}{\eta} \, m^{\frac{C}{\eta}} \, \frac{1}{k^{1+\frac{C}{\eta}}} =$$

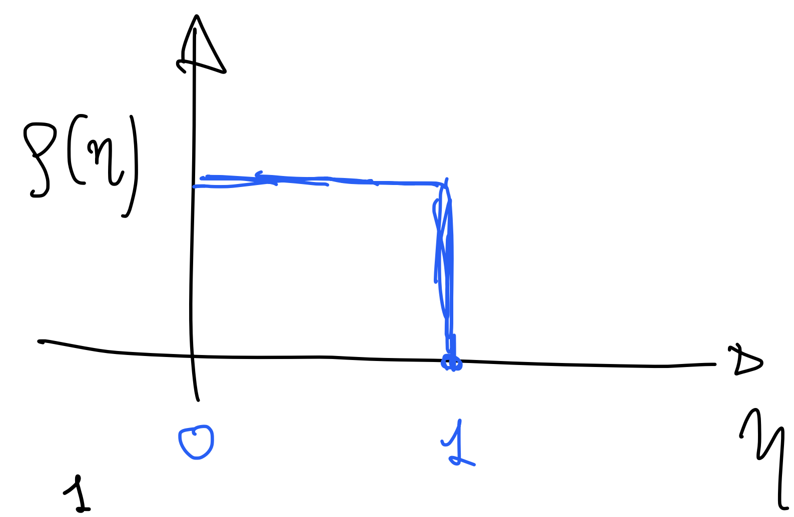
$$= \int d\eta \, \delta(\eta - 1) \, \frac{2}{\eta} \, m^{\frac{2}{\eta}} \, \frac{1}{k^{1+\frac{2}{\eta}}} = \frac{2}{1} \, m^{\frac{2}{1}} \, \frac{1}{k^{1+\frac{2}{1}}} =$$

$$= 2m^2 \frac{1}{k^3} = \frac{2m^2}{k^3}$$

which coincides with ⊛ ⊛ ⊛ of BA model $(\gamma = 3)$

$\boxed{\text{LIMIT CASE 2}}$ UNIFORM DISTRIBUTION OF FITNESS

$\rho(\eta) = 1$ for $\eta \in [0,1]$



C satisfies

$$1 = \int d\eta \, \frac{\rho(\eta)}{\frac{C}{\eta} - 1} = \int_0^1 d\eta \, \frac{1}{\frac{C}{\eta} - 1} = \int_0^1 d\eta \, \frac{\eta}{C - \eta} =$$

$$= \Big[-\eta - C \ln(-C + \eta) \Big]_0^1 =$$

Check Derivative $= -1 - C \frac{1}{-C+\eta} = \frac{\cancel{C} - \eta - \cancel{C}}{-C+\eta} = \frac{\eta}{C - \eta}$

$$= -1 - c\ln(-c+1) + c\ln(-c) =$$

$$= -1 + c\ln\frac{-c}{-c+1}$$

Hence $$1 = -1 + c\ln\frac{c}{c-1}$$

$$\frac{2}{c} = \ln\frac{c}{c-1}$$

$$e^{-\frac{2}{c}} = 1 - \frac{1}{c}$$

which has solution $c^* \simeq 1.255..$

$$P(k) = \int_0^1 d\eta\, \frac{c^*}{\eta}\, m^{\frac{c^*}{\eta}} \frac{1}{k^{\frac{c^*}{\eta}+1}} \simeq \frac{k^{-(1+c^*)}}{\ln k} = \frac{k^{-2.255}}{\ln k}$$

by SADDLE POINT method

(see Ginestra's lecture notes)

Power-law exponent

$\gamma = 2.255..$