WEEK 10 Lecture 1

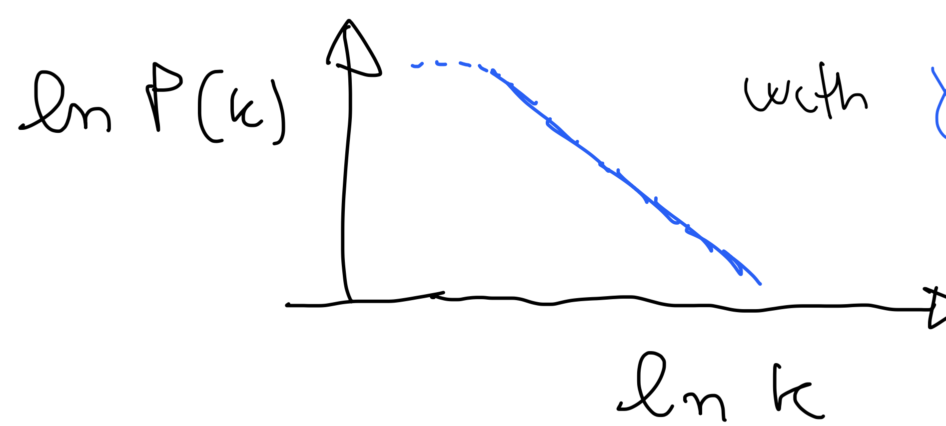
CHAPTER 6
EVOLVING NETWORKS

In the last 2 weeks:

- SCALE-FREE NETWORKS

$$P(k) \simeq C_1 k^{-\gamma} \quad \text{for } k \gg 1$$

with $\gamma \in (2, 3]$



As $N \to \infty$ $\begin{cases} \langle k \rangle \to \text{finite constant} \\ \langle k^2 \rangle \to \infty \end{cases}$

— MODELS OF GROWING NETWORKS

BA MODEL

Preferential attachment

$$\Pi_i = \frac{k_i}{\sum_j k_j} \longrightarrow \text{SCALE-FREE } P(k) \text{ with } \gamma = 3$$

UNIFORM ATTACHMENT MODEL

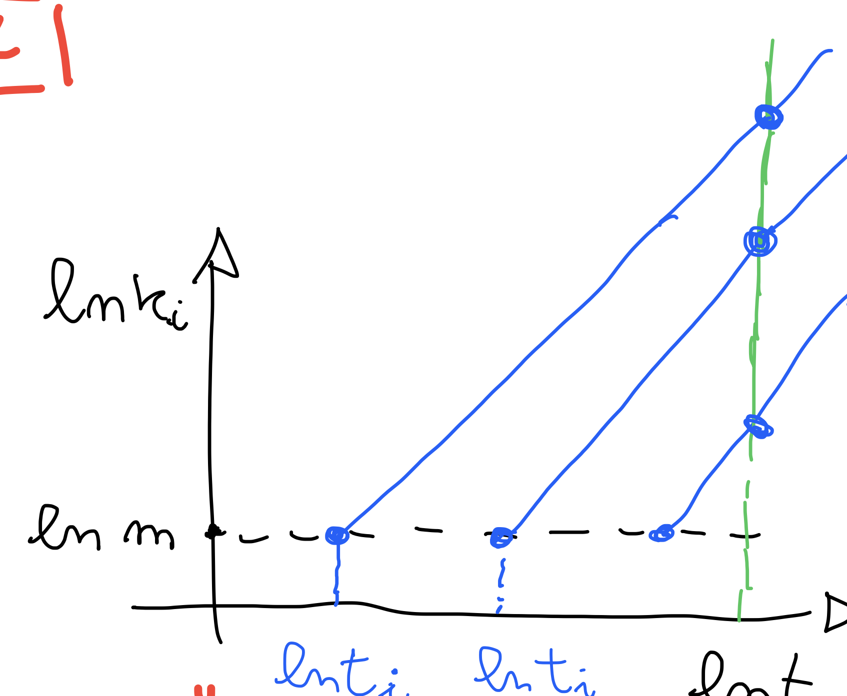
$$\Pi_i = \frac{1}{N(t)} \quad \forall i \longrightarrow \text{EXPONENTIAL } P(k)$$

6.2 THE BIANCONI-BARABASI MODEL

— In the BA model

$$K_i(t) = m \left(\frac{t}{t_i}\right)^{\frac{1}{2}}$$

$$t \geq t_i$$



"First mover advantage" or "RICH gets RICHER" mechanism

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— Paper published in EPL 2001 → Intrinsic fitness of a node

DEF BB MODEL

At time $t = 1$ we start with a network of $\begin{cases} m_0 \text{ nodes} \\ m_0 \text{ links} \end{cases}$

Each node i of the network is assigned a FITNESS η_i drawn from a distribution $\rho(\eta)$

At each time $t > 1$

1. A new node with m new links is added to the network
2. Every new link is attached to an existing node i with a probability

preferential attachment

$$\boxed{\Pi_i = \frac{\eta_i \cdot K_i}{\sum_j \eta_j K_j}}$$

degree of node i

fitness of node i

NOTICE: given two nodes with the same degree, the node with higher fitness is more likely to acquire new links

PROPOSITION $K_i(t)$

In the MF approximation the degree $k_i(t)$ of a node i arrived in the network at time t_i is given by:

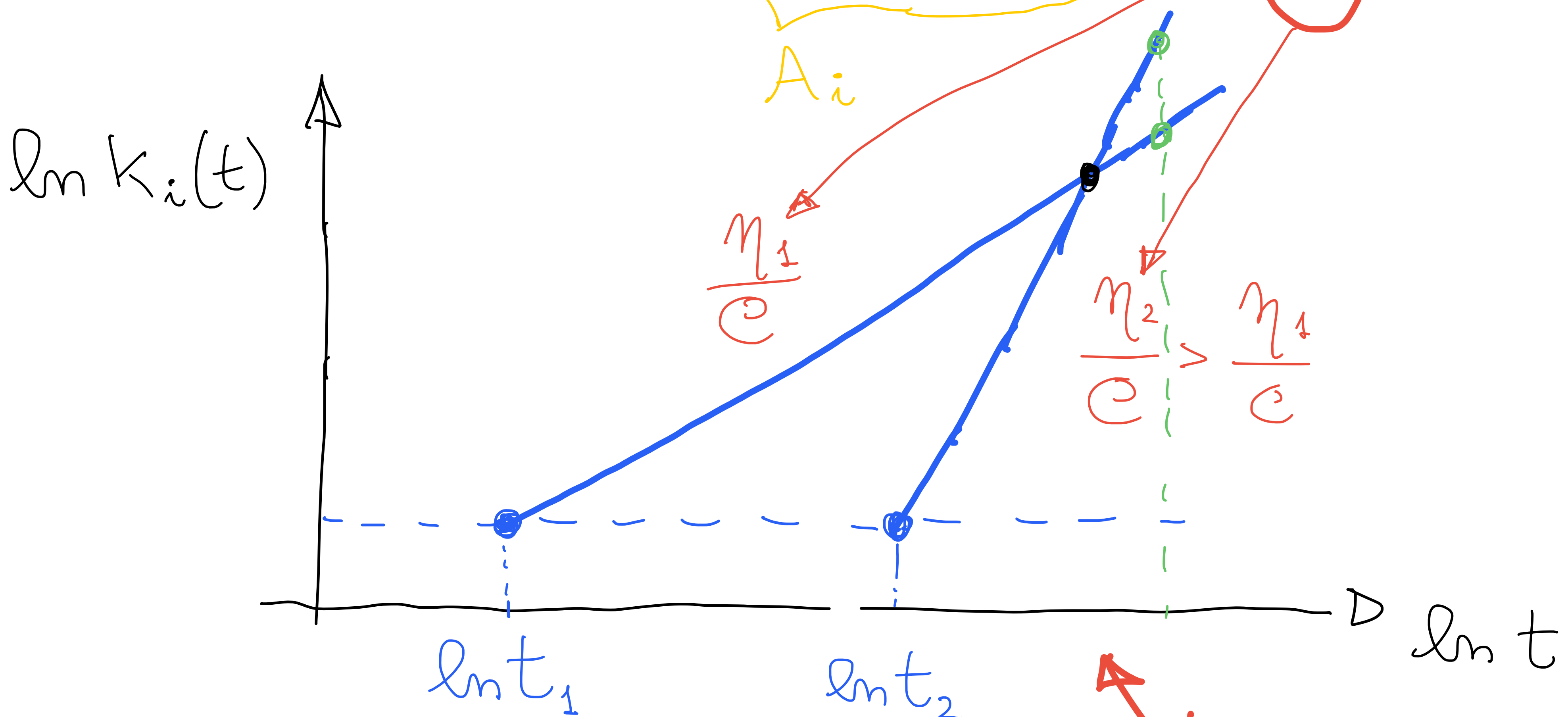
$$\boxed{K_i(t) = m\left(\frac{t}{t_i}\right)^{\frac{\eta_i}{C}} \quad \text{for } t \geq t_i}$$

where C is a constant which depends on the fitness distribution $\rho(\eta)$ and satisfies

$$\boxed{\int \frac{\rho(\eta)}{\frac{C}{\eta} - 1} \, d\eta = 1}$$

we still have a power-law dependence of k_i on t, but now with an exponent $\frac{\eta_i}{C}$ which depends on the node fitness

$$\ln k_i(t) = \underbrace{\ln m - \frac{\eta_i}{c}\ln t_i}_{A_i} + \frac{\eta_i}{c}\ln t$$



"FIT gets RICHER" mechanism

"Also latecomers can make it"

Proof In the MF approximation $k_i(t)$ satisfies

$$\begin{cases} \dfrac{dk_i(t)}{dt} = \tilde{\Pi}_i & \text{for } t > t_i \\ k_i(t=t_i) = m \end{cases}$$

expected increase in the # of links of node i at time t

usual initial condition

$$\tilde{\Pi}_i = m\,\Pi_i = m \frac{\eta_i k_i}{\sum_j \eta_j k_j}$$

Hence

$$\boxed{\begin{cases} \dfrac{dk_i}{dt} = m \dfrac{\eta_i k_i}{\sum_j \eta_j k_j} & \text{for } t > t_i \\ k_i(t_i) = m \end{cases}}$$

MF equation for the BB model

SELF-CONSISTENT SOLUTION

(a) We make an assumption (to simplify the expression of the normalization)

(b) We solve the equation under this assumption

(c) Finally we check that the solution is consistent with the assumption

ⓐ The assumption we make is:

$$\sum_j \eta_j k_j \simeq m C t \qquad t \gg 1$$

Therefore $\displaystyle\lim_{t\to\infty} \frac{\sum_j \eta_j k_j}{m t} = C$ where C is a constant independent on the network realisation (on the sampling of η from $\rho(\eta)$)

ⓑ Under this assumption we have:

$$\frac{dk_i}{dt} = \cancel{m} \frac{\eta_i k_i}{\cancel{m} C t} \quad \text{for } t > t_i \quad t \gg 1$$

Integrating between

$$\int_{k_i(t_i)=m}^{k_i(t)} \frac{dk_i'}{k_i'} = \frac{\eta_i}{C} \int_{t_i}^{t} \frac{dt'}{t'}$$

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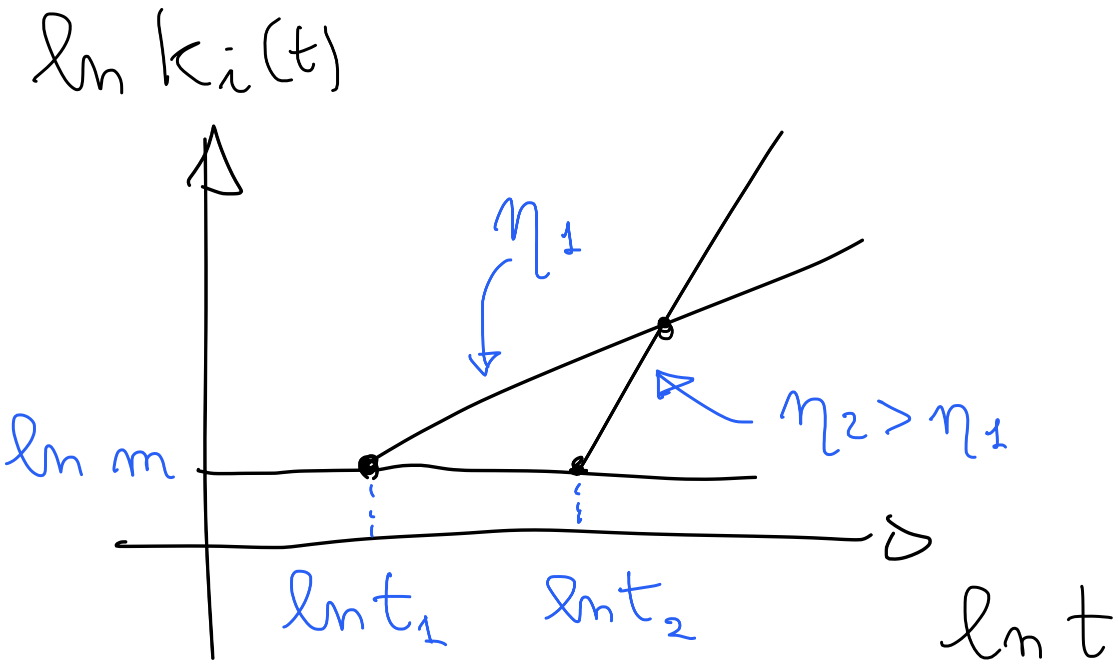
$$\ln \frac{k_i(t)}{m} = \frac{\eta_i}{C} \ln \frac{t}{t_i}$$

$$\boxed{k_i(t) = m \left(\frac{t}{t_i}\right)^{\frac{\eta_i}{C}} \qquad t \geq t_i, \quad t \gg 1}$$

which is with exponent

$$\frac{\eta_i}{C} \leq 1 \qquad \left(\text{typically } \frac{\eta_i}{C} < 1\right)$$

Since we add m links at each time, the degree of a node cannot increase in time faster than linearly



ⓒ We need to check that the assumption

$$\lim_{t\to\infty} \frac{\sum_j \eta_j k_j}{mt} = C \quad \text{with } C \text{ constant, i.e. independent from the network realisation}$$

is consistent with the solution

Practically our aim is to calculate C

$$G = \lim_{t \to \infty} \frac{\langle \sum_J \eta_J k_J \rangle}{m t}$$

where $\langle - - - - \rangle$ indicates the average over the network realisations

We observe that:

$$\langle \sum_J \eta_J k_J \rangle = \sum_J \langle \eta_J k_J \rangle$$

In the MF approximation $\sum_J \longrightarrow \int dt_J$

Hence

we are at a time t

$$\sum_J \langle \eta_J k_J \rangle = \int_1^t dt_J \langle \eta_J k_J \rangle$$

$$\langle \eta_J k_J \rangle = \int d\eta \, \rho(\eta) \, \eta \cdot k(t \mid t_J, \eta_J = \eta)$$

given that the node was introduced at time t_J and has fitness η

We can use

$$k(t \mid t_J, \eta_J = \eta) = m \left(\frac{t}{t_J} \right)^{\frac{\eta}{C}}$$

Putting all the pieces together we have:

$$\sum_J \langle \eta_J k_J \rangle = \int_1^t dt_J \int d\eta\, \rho(\eta)\, \eta \cdot m \left(\frac{t}{t_J}\right)^{\frac{\eta}{c}} =$$

$$= m \int d\eta\, \rho(\eta)\, \eta\, t^{\frac{\eta}{c}} \int_1^t dt_J\, t_J^{-\frac{\eta}{c}} =$$

$$= m \int d\eta\, \rho(\eta)\, \eta \cdot t^{\frac{\eta}{c}} \left. \frac{t_J^{1-\frac{\eta}{c}}}{1-\frac{\eta}{c}} \right|_1^t =$$

$$= m \int d\eta\, \frac{\eta\, \rho(\eta)}{1-\frac{\eta}{c}} \cdot t^{\frac{\eta}{c}} \left[t^{1-\frac{\eta}{c}} - 1 \right] =$$

$$= m \int d\eta\, \frac{\eta \cdot \rho(\eta)}{1-\frac{\eta}{c}} \left[t - \cancel{t^{\frac{\eta}{c}}} \right] \simeq$$

For large times

$t \gg t^{\frac{\eta}{c}}$ because $\frac{\eta}{c} < 1$

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Finally

$$C = \lim_{t\to\infty} \frac{\sum_J \langle \eta_J k_J \rangle}{mt} \simeq \lim_{t\to\infty} \frac{\cancel{mt} \int d\eta \, \frac{\eta \cdot \rho(\eta)}{1-\frac{\eta}{C}}}{\cancel{mt}} =$$

$$= \int d\eta \, \rho(\eta) \frac{\eta}{1-\frac{\eta}{C}}$$

So we got a self-consistent equation for the constant C:

$$C = \int d\eta \, \rho(\eta) \frac{\eta}{1-\frac{\eta}{C}}$$

Now dividing by C

$$1 = \int d\eta \frac{\rho(\eta)}{\frac{C}{\eta}\left(1-\frac{\eta}{C}\right)} = \int d\eta \frac{\rho(\eta)}{\frac{C}{\eta}-1}$$

which gives

$$\boxed{1 = \int d\eta \frac{\rho(\eta)}{\frac{C}{\eta}-1}}$$

that is the result in the proposition

To conclude, as long as this equation has solution for C_i we have solved the BB model

We will show this for some choices of the distribution $\rho(\eta)$

In the next lecture we will also derive the expression for $P(k)$ in the BB model.

Do not forget to work and submit ASSESSED COURSEWORK 5

(opens on FRIDAY this week)