Queen Mary
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Complex Networks (MTH6142)
Solutions of Formative Assignment 7

- **Growing network model**

  Consider the following model for a growing simple network.
  We adopt the following notation: N and L indicate respectively the total number of nodes and links of the network, A_{ir} indicates the generic element of the adjacency matrix $\mathbf{A}$ of the network and k_i indicates the degree of node i.
  At time $t = 0$ the network is formed by a $n_0 = 2$ nodes and a single link (initial number of links $m_0 = 1$) connecting the two nodes.
  At every time step $t > 0$ the network evolve according to the following rules:

  - A single new node joins the network.
  - An existing link (i, r) between a node i and a node r (two nodes of the network) is chosen randomly with uniform probability

  $$\pi_{(i,r)} = \frac{A_{ir}}{L}$$

  and the new node is linked to both node i and node r.

  a) Show that in this network evolution at each time step the average number of links $\tilde{\Pi}_i$ added to node i follows the preferential attachment rule, i.e.

  $$\tilde{\Pi}_i = \sum_{r=1}^{N} \pi_{(i,r)} = 2\frac{k_i}{\sum_{j=1}^{N} k_j}.$$

  b) What is the total number of links in the network at time t? What is the total number of nodes?

  c) What is the average degree $\langle k \rangle$ of the network at time t?

  d) Use the result at point a) to derive the time evolution $k_i = k_i(t)$ of the average degree k_i of a node i for $t \gg 1$ in the mean-field, continuous approximation.

  e) What is the degree distribution of the network at large times in the mean-field approximation?

f) Let $N_k(t)$ be the average number of nodes with degree k at time t. Write the master equation for $N_k(t)$.

g) Solve the master equation, finding the exact result for the degree distribution $P(k)$ in the limit $N \to \infty$.

- *Notes on solution*
a) The average number of links $\tilde{\Pi}_i$ added to node i in timestep t is given by

$$\tilde{\Pi}_i = \sum_{r=1}^{N} \pi_{(i,r)} = \sum_{r=1}^{N} \frac{A_{ir}}{L}. \tag{1}$$

Using

$$\begin{aligned} L &= \frac{1}{2}\sum_{j=1}^{N} k_j \\ k_i &= \sum_{r=1}^{N} A_{ir} \end{aligned} \tag{2}$$

we get

$$\tilde{\Pi}_i = \frac{2k_i}{\sum_{j=1}^{N} k_j} \tag{3}$$

b) Since initially the number of links is $m_0 = 1$ and at each time we add two links we have $L = 1 + 2t$. Since initially we have $n_0 = 2$ nodes and at each time we add a node we have $N = 2 + t$. c)The average degree $\langle k \rangle$ is given by

$$\langle k \rangle = \frac{2L}{N} = \frac{2(1+2t)}{2+t}. \tag{4}$$

d)In the mean-field approximation, the degree $k_i(t)$ of node i at time t satisfies the following differential equation

$$\frac{dk_i(t)}{dt} = \tilde{\Pi}_i = \frac{2k_i}{\sum_{j=1}^{N} k_j} \tag{5}$$

In the limit $t \gg 1$, we have $\sum_j k_j = 2L \simeq 4t$. Therefore we can write the dynamical mean-field equation for the degree $k_i(t)$ of node i, getting

$$\frac{dk_i}{dt} = \frac{2k_i}{4t} = \frac{k_i}{2t}, \tag{6}$$

with initial condition $k_i(t_i) = 2$.

This equation has solution

$$k_i(t) = 2\left(\frac{t}{t_i}\right)^{1/2}. \tag{7}$$

e) The probability $P(k_i(t) > k)$ that a random node has degree $k_i(t) > k$, in the mean-field approximation can be calculated as follows

$$P(k_i(t) > k) = P\left(2\left(\frac{t}{t_i}\right)^{(1/2)} > k\right) = P\left(t_i < t\left(\frac{2}{k}\right)^2\right) = \left(\frac{2}{k}\right)^2. \tag{8}$$

Therefore, the degree distribution $P(k)$ is given by

$$P(k) = \frac{dP(k_i(t) < k)}{dk} = -\frac{d}{dk}\left(\frac{2}{k}\right)^2 = \left(\frac{2}{k}\right)^3. \tag{9}$$

f) The master equation for $N_k(t)$ reads

$$N_k(t+1) = N_k(t) + \frac{k-1}{2t}N_{k-1}(t)(1-\delta_{k,2}) - \frac{k}{2t}N_k(t) + \delta_{k,2} \tag{10}$$

g) Assuming $N_k(t) \simeq (t + n_0)P(k)$ for large t
1 we get

$$P(k) = \frac{k-1}{2}P(k-1)(1-\delta_{k,2}) - \frac{k}{2}P(k) + \delta_{k,2}. \tag{11}$$

giving

$$P(k) = \frac{k-1}{k+2}P(k-1) \tag{12}$$

for $k > 2$ and

$$P(2) = \frac{1}{2}. \tag{13}$$

Solving these equations we get

$$P(k) = \frac{12}{k(k+1)(k+2)}. \tag{14}$$