WEEK 9 Tutorial

$\boxed{\text{FA 7}}$ model of growing networks

FEEDBACK ON ASSESSED COURSEWORK 4

The reproduction number R of an epidemic spreading process taking place on a random network with degree distribution $P(k)$ is given by

$$R = \lambda \frac{\langle k(k-1)\rangle}{\langle k\rangle},$$

where k indicates the degree of the nodes and the average $\langle \ldots \rangle$ indicates the average over the degree distribution, $P(k)$.

REPRODUCTION NUMBER

expected # of cases generated by one case

$$R = \lambda \frac{\langle k(k-1)\rangle}{\langle k\rangle}$$

DISEASE infectivity of the virus

$\frac{1}{4}$

NETWORK with a given $P(k)$

"BRANCHING RATIO"

When $R > 1$ the infection will spread

1

Consider an epidemics with infectivity $\lambda = 1/4$. Investigate how the network topology can determine the regime of the epidemics in the following cases.

(A) Consider the following two cases: a Poisson network with average degree $c = 3$ and a Poisson network with average degree $c = 5$. Evaluate R, and establish in which regime the epidemic process is in these two networks. **[1 MARK]**

(B) Evaluate R for a scale-free network with degree distribution $P(k) = Ck^{-\gamma}$, minimum degree m, maximum degree K and power-law exponent $\gamma = 2.5$ using the continuous approximation for the degrees. **[2 MARKS]**

(C) Take the scale-free network considered in point (B), calculate R and establish in which regime the epidemic process is if $m = 2$, $K = 50$. **[1 MARK]**

A

Poisson networks $\to$ $c = \langle k \rangle = 3$; $c = \langle k \rangle = 5$

$$P_P(k) = \frac{c^k}{k!} e^{-c} \qquad k = 0, 1, 2, \dots$$

POISSON DEGREE DISTRIBUTION

From W5

$$\langle k \rangle = c$$

$$\langle k(k-1) \rangle = c^2$$

From GENERATING FUNCTION

$$G_P(x) = e^{cx} \cdot e^{-c}$$

2

$$\sigma^2 = \langle k^2 \rangle - \langle k \rangle^2 = c \qquad G_p^{[m]}(1) = \langle k(k-1)\dots(k-m+1)\rangle$$

$$\sigma = \sqrt{c}$$

So we directly know that $\begin{cases} \langle k(k-1)\rangle = c^2 \\ \langle k \rangle = c \end{cases}$

$$R = \lambda \frac{\langle k(k-1)\rangle}{\langle k \rangle} = \lambda \frac{c^2}{c} = \lambda c$$

CASE $c=3$ $\quad R = \lambda c = \frac{1}{4} \cdot 3 = \frac{3}{4} < 1 \Rightarrow$ SUBCRITICAL REGIME

CASE $c=5$ $\quad R = \lambda c = \frac{1}{4} \cdot 5 = \frac{5}{4} > 1 \Rightarrow$ SUPERCRITICAL

B $\quad P(k) = C k^{-\gamma} \qquad \gamma = 2.5$

min degree m

Max degree K

$$\frac{\langle k(k-1)\rangle}{\langle k \rangle} = \frac{\langle k^2 - k\rangle}{\langle k \rangle} = \frac{\langle k^2\rangle - \langle k \rangle}{\langle k \rangle} = \frac{\langle k^2\rangle}{\langle k \rangle} - 1$$

We need to calculate $\langle k \rangle$ and $\langle k^2 \rangle$ in the continuous k approximation ← repeat calculations done in W8 (Sect 5.3 : MOMENTS)

$$\langle k \rangle = \int_m^{\bar{K}} k\, P(k)\, dk = C \int_m^{\bar{K}} k^{1-\gamma}\, dk = C \left[\frac{k^{2-\gamma}}{2-\gamma} \right]_m^{\bar{K}} =$$

$P(k) = C k^{-\gamma}$

$1-\gamma \neq -1 \quad \gamma \neq 2$ ok since $\gamma = 2.5$

$$= \frac{C}{\gamma-2} \left[m^{2-\gamma} - \bar{K}^{2-\gamma} \right] =$$

$\gamma = 2.5$

$$= \frac{C}{0.5} \left[m^{-0.5} - \bar{K}^{-0.5} \right]$$

$$\langle k^2 \rangle = \int_m^{\bar{K}} k^2\, P(k)\, dk = C \int_m^{\bar{K}} k^{2-\gamma}\, dk = C \left[\frac{k^{3-\gamma}}{3-\gamma} \right]_m^{\bar{K}} =$$

$2-\gamma \neq -1$

$\gamma \neq 3$ ← $\gamma = 2.5$

$$= \frac{C}{3-\gamma}\left[K^{3-\gamma} - m^{3-\gamma}\right] = \frac{C}{0.5}\left[K^{0.5} - m^{0.5}\right]$$

$$R = \lambda\left[\frac{\langle k^2\rangle}{\langle k\rangle} - 1\right] = \lambda\left[\frac{\cancel{\frac{C}{0.5}}\left(K^{0.5} - m^{0.5}\right)}{\cancel{\frac{C}{0.5}}\left(m^{-0.5} - K^{-0.5}\right)} - 1\right] =$$

$$= \lambda\left[\frac{K^{0.5} - m^{0.5}}{\frac{1}{m^{0.5}} - \frac{1}{K^{0.5}}} - 1\right] = \lambda\left[\frac{\cancel{K^{0.5} - m^{0.5}}}{\frac{\cancel{K^{0.5} - m^{0.5}}}{m^{0.5} \cdot K^{0.5}}} - 1\right] =$$

$$= \lambda\left[m^{0.5} \cdot K^{0.5} - 1\right] = \lambda\left[\sqrt{m \cdot K} - 1\right]$$

c) For $m=2$, $K=50$, $\lambda = \frac{1}{4}$ we get

$$R = \frac{1}{4}\left[\sqrt{2 \cdot 50} - 1\right] = \frac{1}{4}\left[\sqrt{100} - 1\right] = \frac{9}{4} = 2.25 > 1$$

SUPERCRITICAL

5

$\boxed{\text{FA 7}}$ A model of growing network

Will be covered in tutorial of W 11