WEEK 9 Lecture 2

5.6 GROWING MODEL WITH UNIFORM ATTACHMENT

DEF THE UNIFORM ATTACHMENT MODEL

At time $t=1$ we start with a network of $\begin{cases} m_0 & \text{nodes} \\ m_0 & \text{links} \end{cases}$

At each time $t>1$

1. A new node with m links is attached to the network
2. Every new link is attached to an existing node i with a uniform probability

$$\Pi_i = \frac{1}{N(t)}$$

$N(t)$: # of nodes in the network at time t

PROPOSITION

In the MF approximation the degree $k_i(t)$ of a node i arrived in the network at time t_i is given by:

$$k_i(t) = m + m \ln\left(\frac{t}{t_i}\right) \quad \text{for } t \geq t_i$$

Proof

$$\tilde{\Pi}_i = m \Pi_i = \frac{m}{N(t)}$$

$\tilde{\Pi}_i$: expected increase in the # of links of node i at time t

Π_i: probability that a new link connects to i

$$\begin{cases} \dfrac{dk_i}{dt} = \dfrac{m}{N(t)} & \text{for } t > t_i \\ k_i(t_i) = m \end{cases}$$

MF equations

Let us observe that: $N(t) = m_0 + (t-1) \simeq t \quad \text{for } t \gg 1$

The MF equation becomes

$$\frac{dk_i}{dt} = \frac{m}{t} \quad \text{for } t >> 1$$

Integrating between t_i and t

$$\int_{k_i(t_i)=m}^{k_i(t)} dk_i = m \int_{t_i}^{t} \frac{dt'}{t'}$$

$$k_i(t) - m = m \ln \frac{t}{t_i}$$

$$k_i(t) = m + m \ln \frac{t}{t_i} \quad \text{for } t \geq t_i, \; t >> 1$$

which is what we wanted to prove

PROPOSITION

In the **MF** approximation the degree distribution of the UNIFORM ATTACHMENT MODEL is in the limit $t \to \infty$

$$P(k) = \frac{e}{m} e^{-\frac{k}{m}}$$

Proof

$$\text{Prob}\left(k_i(t) > k\right) = \text{Prob}\left(m + m \ln \frac{t}{t_i} > k\right) =$$

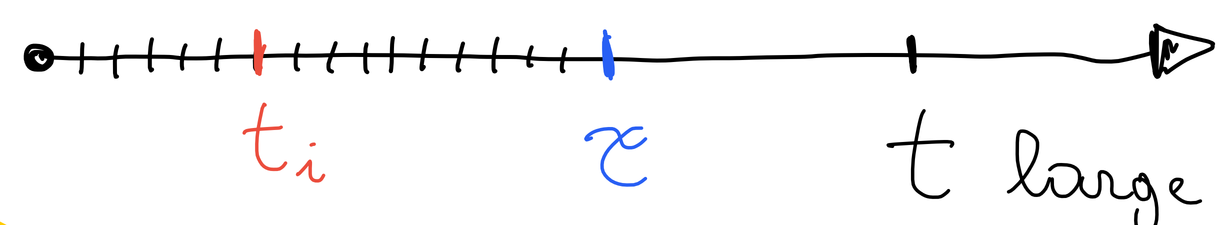
$$k_i(t) = m + m \ln \frac{t}{t_i}$$

$$m + m \ln \frac{t}{t_i} > k \qquad \ln \frac{t}{t_i} > \frac{k}{m} - 1 \qquad \frac{t}{t_i} > e^{\frac{k}{m} - 1}$$

$$\frac{t_i}{t} < e^{-\frac{k}{m} + 1} \qquad t_i < t\, e^{-\frac{k}{m} + 1}$$

$$= \text{Prob}\left(t_i < \underbrace{t\, e^{-\frac{k}{m} + 1}}_{\tau}\right)$$

Now I can use the same formula: $\text{Prob}(t_i < \tau) \simeq \frac{\tau}{t}$



$$\text{Prob}\left(k_i(t) > k\right) = \text{Prob}\left(t_i < \underbrace{t\, e^{-\frac{k}{m} + 1}}_{\tau}\right) \simeq \frac{t\, e^{-\frac{k}{m} + 1}}{t} = e^{-\frac{k}{m} + 1}$$

4

$$\text{Prob}\left(k_i(t) > k\right) \simeq e^{-\frac{k}{m}+1}$$

The degree distribution $P(k)$ can be obtained

$$P(k) = \frac{d}{dk}\,\text{Prob}\left(k_i(t) \leq k\right)$$

$$= \frac{d}{dk}\left[1 - \text{Prob}\left(k_i(t) > k\right)\right] \simeq \frac{d}{dk}\left[1 - e^{-\frac{k}{m}+1}\right] =$$

$$= \frac{1}{m}\, e^{-\frac{k}{m}+1}$$

$$P(k) \simeq \frac{e}{m}\, e^{-\frac{k}{m}} \quad \text{for } k \geq m$$

We can now use the Master Equation approach to get $P(k)$

PROPOSITION

The degree distribution $P(k)$ of the UNIFORM ATTACHMENT MODEL in the limit $t \to \infty$ $(N \to \infty)$ is given by

$$P(k) = \left(\frac{m}{1+m}\right)^{k-m} \cdot \frac{1}{1+m} \qquad \text{for } k \geq m$$

$P(k)$ is decaying exponentially

For $t \gg 1$ $\quad N(t) \simeq t \quad$ and $\quad \tilde{\Pi}(k) = \frac{m}{N(t)} \simeq \frac{m}{t}$

Inserting this in the MASTER EQUATION:

$$\begin{cases} N_k(t+1) = N_k(t) + \tilde{\Pi}(k-1)\, N_{k-1}(t) - \tilde{\Pi}(k)\, N_k(t) & k > m \\ N_k(t+1) = N_k(t) + 1 - \tilde{\Pi}(k)\, N_k(t) & k = m \end{cases}$$

We get

$$\begin{cases} N_k(t+1) = N_k(t) + \frac{m}{t} N_{k-1}(t) - \frac{m}{t} N_k(t) & k > m \\ N_k(t+1) = N_k(t) + 1 - \frac{m}{t} N_k(t) & k = m \end{cases}$$

For $t >> 1$ we assume that $N_k(t) = N\,P(k) \simeq t\,P(k)$

$N(t) \simeq t$

$$\begin{cases} (t+1)\,P(k) = t\,P(k) + \frac{m}{t}\,t\,P(k-1) - \frac{m}{t}\,t\,P(k) & k > m \\ (t+1)\,P(k) = t\,P(k) + 1 - \frac{m}{t}\,t\,P(k) & k = m \end{cases}$$

$$\begin{cases} (1+m)\,P(k) = m\,P(k-1) & k > m \\ (1+m)\,P(k) = 1 & k = m \end{cases}$$

$$\begin{cases} P(k) = \frac{m}{1+m} P(k-1) & k > m \\ P(m) = \frac{1}{1+m} \end{cases}$$

We can solve the Master equation recursively

$$P(k) = \frac{m}{1+m} P(k-1) = \frac{m}{1+m} \frac{m}{1+m} P(k-2) = \dots$$

$$= \left(\frac{m}{1+m}\right)^{k-m} P(m)$$

$$P(k) = \left(\frac{m}{1+m}\right)^{k-m} \cdot \frac{1}{1+m} \qquad k \geq m$$

which is the exponential distribution we wanted to find

Furthermore we can rewrite this P(k) in a way that looks closer to the result in MF

We use $y = a^x \iff y = e^{\ln a^x} = e^{x \ln a}$

$x = k-m$

$a = \frac{m}{1+m}$

$$P(k) = \frac{1}{1+m} e^{(k-m)\ln\frac{m}{1+m}} = \frac{1}{1+m} e^{m \ln\left(1+\frac{1}{m}\right)} e^{-k \ln\left(1+\frac{1}{m}\right)}$$

Constant

Furthermore for $m >> 1$, since $\ln\left(1+\frac{1}{m}\right) \simeq \frac{1}{m}$

we have :

$$P(k) \simeq \frac{1}{1+m} \, e^{m \frac{1}{m}} \, e^{-k \frac{1}{m}} = \frac{e}{m} \, e^{-\frac{k}{m}}$$

$m >> 1$