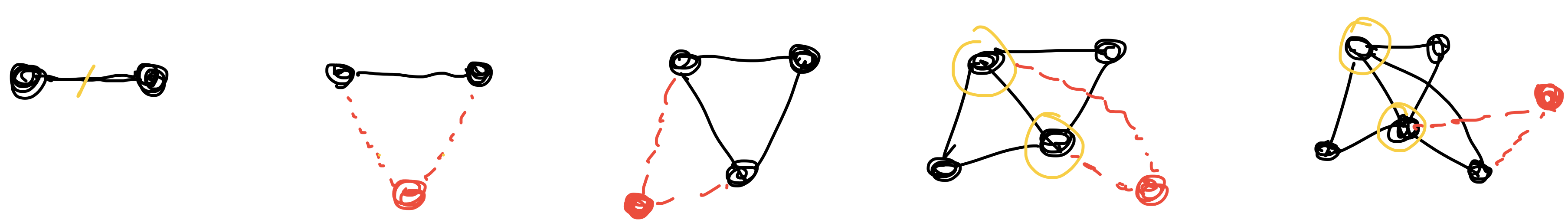
WEEK 9 Lecture 1

Last week we have introduced our first model of network growth the BA model with PREFERENTIAL ATTACHMENT

$$\boxed{\Pi_i = \frac{k_i}{\sum_j k_j}}$$

Visualisation in the case $m_0 = 2$ $m_0 = 1$ $m = 2$

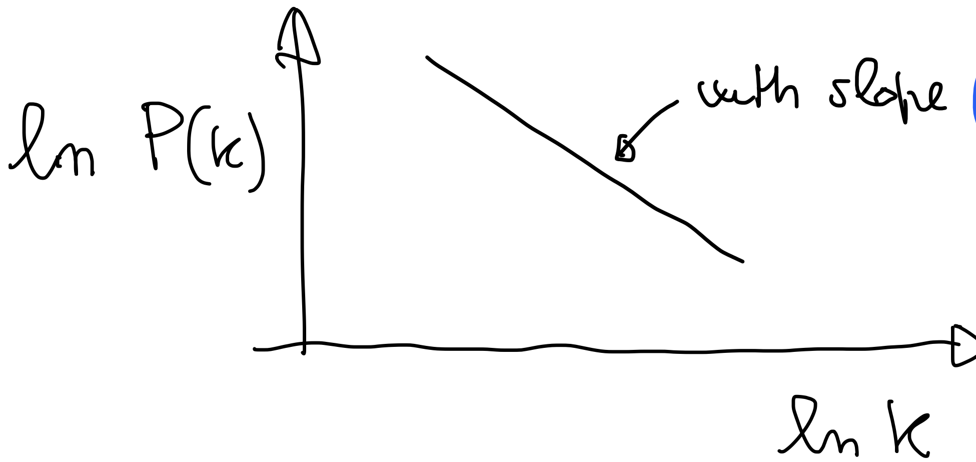
$t = 1$ $t = 2$ $t = 3$ $t = 4$ $t = 5$



Last time we demonstrated that (in the MF approximation)

$$\boxed{k_i(t) = m\left(\frac{t}{t_i}\right)^{\frac{1}{2}} \quad \text{for } t \geq t_i,\ t \gg 1}$$

Today we are going to show the BA model is able to produce scale-free degree distributions



with slope $\gamma = 3$ $\longrightarrow$ $\langle k \rangle \longrightarrow$ finite

$\langle k^2 \rangle \longrightarrow \infty$

as $N \to \infty$

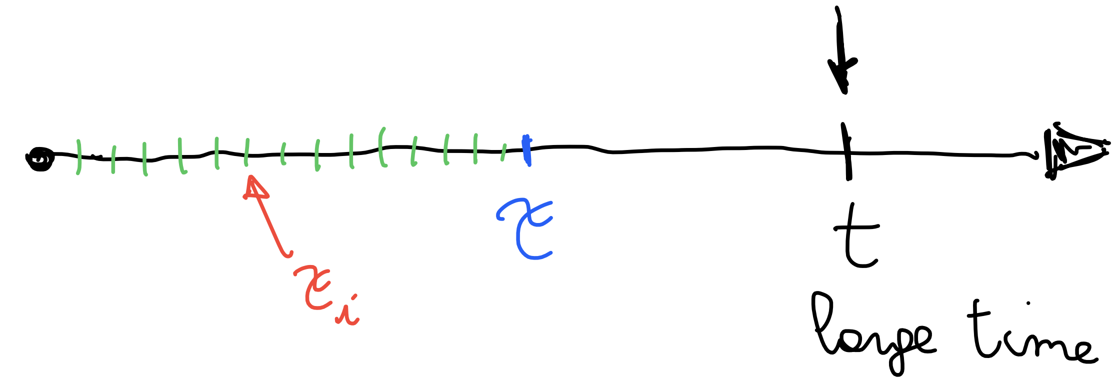
PROPOSITION

In the MF approximation the degree distribution of the BA model is given, in the limit $t \to \infty$, by:

$$\boxed{P(k) \simeq \frac{2m^2}{k^3} = c\, k^{-\gamma} \text{ with } \gamma = 3} \quad (*)$$

Proof At each time step a new node arrives in the network. Hence for $t \gg 1$ the probability that a randomly chosen node has arrived at a time $t_i < \tau$ can be approximated as:

$$\boxed{\mathrm{Prob}(t_i < \tau) \simeq \frac{\tau}{t}}$$



Let us write

$$\mathrm{Prob}\big(k_i(t) > k\big) = \mathrm{Prob}\left(m\left(\frac{t}{t_i}\right)^{\frac{1}{2}} > k\right) =$$

using ⊛⊛ $k_i(t) = m\left(\frac{t}{t_i}\right)^{\frac{1}{2}}$

$$= \mathrm{Prob}\left(\sqrt{\frac{t}{t_i}} > \frac{k}{m}\right) = \mathrm{Prob}\left(\frac{t}{t_i} > \frac{k^2}{m^2}\right) = \mathrm{Prob}\left(t_i < t\,\frac{m^2}{k^2}\right)$$

Hence

$$\mathrm{Prob}\big(k_i(t) > k\big) = \mathrm{Prob}\left(t_i < \underbrace{t\,\frac{m^2}{k^2}}_{\tau}\right) \simeq \frac{\cancel{t}\,\frac{m^2}{k^2}}{\cancel{t}} = \frac{m^2}{k^2}$$

$$\boxed{\mathrm{Prob}\big(k_i(t) > k\big) \simeq \frac{m^2}{k^2}}$$

This is the probability that, if we observe the network a time t, a randomly chosen node has a degree larger than the value k

The degree distribution $P(k)$, i.e. the probability that a randomly chosen node i has degree k, can be written as

$$P(k) = \frac{d}{dk} \operatorname{Prob}(k_i(t) \le k) =$$

$$= \frac{d}{dk}\left[1 - \operatorname{Prob}(k_i(t) > k)\right] =$$

$$= \frac{d}{dk}\left[1 - \frac{m^2}{k^2}\right] = -m^2 \frac{d}{dk} k^{-2} = \frac{2m^2}{k^3}$$

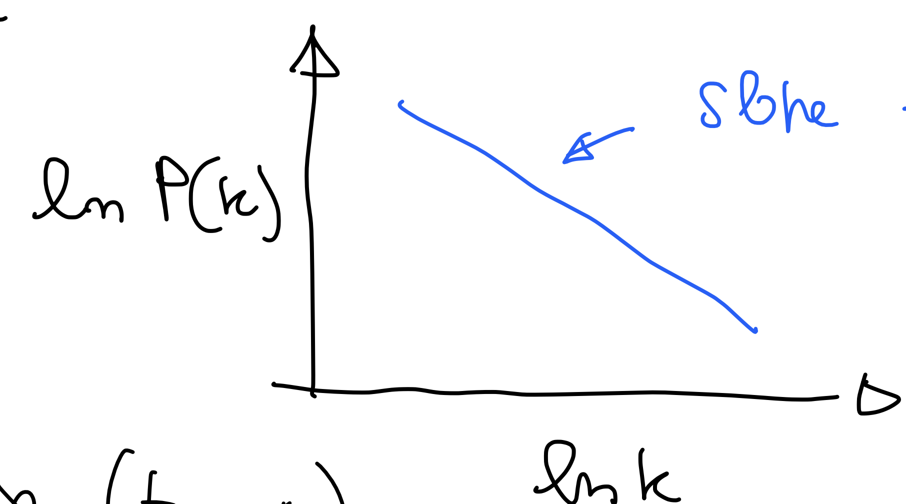
$$P(k) \simeq \frac{2m^2}{k^3} = C\, k^{-\gamma} \quad \text{with } \gamma = 3$$

which is the formula ✱✱✱ we wanted to find

BA model produces SCALE-FREE networks with γ in the range $(2, 3]$ where we have

$\langle k \rangle \longrightarrow$ finite

$\langle k^2 \rangle \longrightarrow \infty$ when $N \to \infty$ $(t \to \infty)$



The M.F. is a strong approximation.

I now present a more rigorous approach that provides exact results for the degree distribution in the limit $t \to \infty$

PROPOSITION **MASTER EQUATION**

The master equation is the equation that describes the evolution of the average number $N_k(t)$ of nodes with degree k at time t.

In the case of the BA model the master equation reads

$$N_k(t+1) = N_k(t) + \tilde{\Pi}(k-1)\, N_{k-1}(t) - \tilde{\Pi}(k)\, N_k(t) \quad \text{for } \boxed{k > m} \quad (**)$$

$$N_k(t+1) = N_k(t) + 1 - \tilde{\Pi}(k)\, N_k(t) \quad \text{for } \boxed{k = m} \quad (**)$$

where $\tilde{\Pi}(k)$ is the probability that a node of degree k acquires a link at time t $\quad \tilde{\Pi}(k) = m \dfrac{k}{\sum_{k'} k' \cdot N_{k'}(t)}$

Proof:

We first observe that

$$\tilde{\Pi}(k) = m\, \underbrace{\Pi_i}_{\substack{\text{probability of linking}\\ \text{to a node } i \text{ of degree } k_i = k}} \Big|_{k_i = k} = m \frac{k}{\sum_J k_J} = m \frac{k}{\sum_{k'} k' N_{k'}(t)}$$

$$\sum_J k_J = \sum_{k'} k' \cdot N_{k'}(t)$$

To derive ⊛⊛ ⊛⊛ we consider separately the cases $\begin{cases} \boxed{k > m} \\ k = m \end{cases}$

$\boxed{k > m}$

$$N_k(t+1) = N_k(t) + \text{GAIN TERM} - \text{LOSS TERM}$$

$$\boxed{\begin{array}{c}\text{GAIN TERM} \\ k-1 \to k\end{array}}$$

$$\underbrace{\tilde{\Pi}(k-1) \cdot N_{k-1}(t)}$$

the average # of nodes of degree $k-1$ at time t that acquire a link at time $t+1$

LOSS TERM

$k \to k+1$

$$\underbrace{\tilde{\Pi}(k) \cdot N_k(t)}$$

average # of nodes of degree k at time t that acquire a link at time t+1

$$N_k(t+1) = N_k(t) + \tilde{\Pi}(k-1)\, N_{k-1}(t) - \tilde{\Pi}(k)\, N_k(t) \qquad k > m$$

$\boxed{k=m}$

$$N_m(t+1) = N_m(t) + \text{GAIN TERM} - \text{LOSS TERM}$$

GAIN TERM

addition of a new node of degree $k=m$

1 ← At each time step one new node of degree m is added to the network

LOSS TERM

$m \to m+1$

$$\underbrace{\tilde{\Pi}(m) \cdot N_m(t)}$$

the average # of nodes of degree m at time t that acquire a new link

$$N_m(t+1) = N_m(t) + 1 - \tilde{\Pi}(m)\cdot N_m(t)$$

or

$$N_k(t+1) = N_k(t) + 1 - \tilde{\Pi}(k)\cdot N_k(t)$$

$k = m$

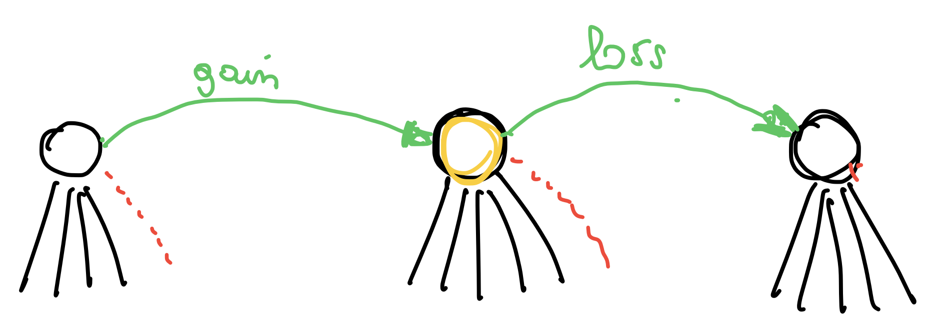
The equations for $k>m$ and $k=m$ can be written in a compact form

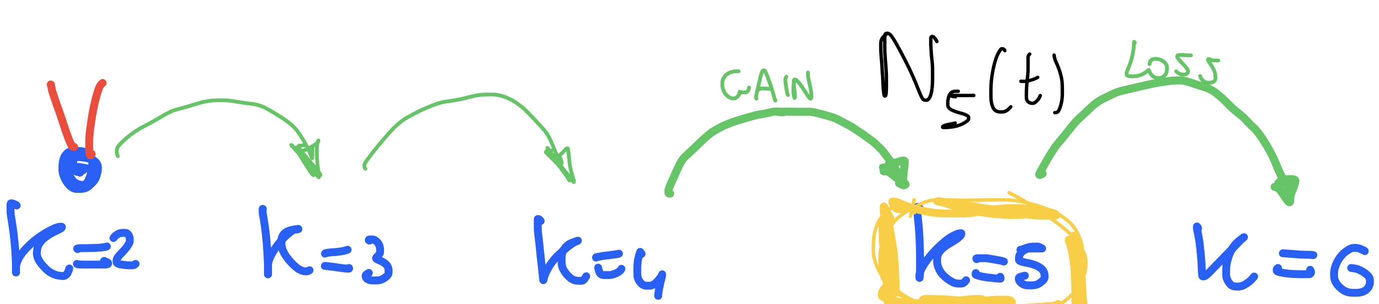
$$N_k(t+1) = N_k(t) + \tilde{\Pi}(k-1)\, N_{k-1}(t) \underbrace{\left[1-\delta_{k,m}\right]}_{1-1} - \tilde{\Pi}(k)\, N_k(t) + \overset{1}{\delta_{k,m}}$$

if $k=m$

where $\delta_{k,m} = \begin{cases} 1 & \text{if } k=m \\ 0 & \text{if } k\neq m \end{cases}$ for $k \geq m$

KRONECKER DELTA





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PROPOSITION $P(k)$

$(N \to \infty)$

The degree distribution $P(k)$ of the BA model, in the limit $t \to \infty$ is given by:

$$\boxed{P(k) = \frac{2m(m+1)}{k(k+1)(k+2)} \quad \text{for } k \geq m}$$

Therefore for $k \gg 1$ $\quad P(k) \simeq C k^{-\gamma}$ with $\gamma = 3$

Proof

$$\tilde{\Pi}(k) = m \frac{k}{\sum_j k_j} = m \frac{k}{\sum_{k'} k' N_{k'}(t)} \simeq \frac{\cancel{m} k}{2 \cancel{m} t} = \frac{k}{2t}$$

For $t \gg 1$ $\quad \sum_j k_j \simeq 2mt$

Hence for $t \gg 1$ the master equation reads:

$$\begin{cases} N_k(t+1) = N_k(t) + \frac{k-1}{2t} N_{k-1}(t) - \frac{k}{2t} N_k(t) & k > m \\ N_m(t+1) = N_m(t) + 1 - \frac{m}{2t} N_m(t) & k = m \end{cases}$$

Assuming that for large times $(t \gg 1)$ the degree distribution converges to a distribution $P(k)$

For $t \gg 1$ $\quad N_k(t) \simeq N\,P(k) \simeq t\,P(k)$ $\quad \boxed{N_k(t) = t\,P(k)}$

at $t \gg 1$ $\quad$ for $t \gg 1$ $\quad N \simeq t$

Hence we can rewrite the master equation in terms of $P(k)$

$$\begin{cases} (\cancel{t}+1)\,P(k) = \cancel{t\,P(k)} + \dfrac{k-1}{2\cancel{t}}\,\cancel{t}\,P(k-1) - \dfrac{k}{2\cancel{t}}\,\cancel{t}\,P(k) & k > m \\ (\cancel{t}+1)\,P(m) = \cancel{t\,P(m)} + 1 - \dfrac{m}{2\cancel{t}}\,\cancel{t}\,P(m) & k = m \end{cases}$$

$$\begin{cases} \left(1 + \dfrac{k}{2}\right) P(k) = \dfrac{k-1}{2}\,P(k-1) \\ \left(1 + \dfrac{m}{2}\right) P(m) = 1 \end{cases}$$

$$P(k) = \frac{k-1}{k+2}\,P(k-1)$$

$$P(m) = \frac{1}{1+\frac{m}{2}} = \frac{2}{m+2}$$

$$\boxed{\begin{cases} P(k) = \dfrac{k-1}{k+2}\,P(k-1) & k > m \\ P(m) = \dfrac{2}{m+2} \end{cases}}$$

$$P(k) = \frac{k-1}{k+2} P(k-1) = \frac{k-1}{k+2}\,\frac{k-2}{k+1} P(k-2) = \frac{k-1}{k+2}\,\frac{k-2}{k+1}\,\frac{k-3}{k} P(k-3) = \dots =$$

$$= \frac{\cancel{k-1}}{k+2}\,\frac{\cancel{k-2}}{k+1}\,\frac{\cancel{k-3}}{k}\,\frac{\cancel{k-4}}{\cancel{k-1}} \cdots \cdots \frac{\cancel{m+3}}{\cancel{m+6}}\,\frac{m+2}{\cancel{m+5}}\,\frac{m+1}{\cancel{m+4}}\,\frac{m}{\cancel{m+3}} P(m)$$

From $P(k) = \frac{k-1}{k+2} P(k-1)$

with $k-1 = m$

$k = m+1$

$k+2 = m+3$

$$P(m+1) = \frac{m}{m+3} P(m)$$

$$P(m+2) = \frac{m+1}{m+4} P(m+1)$$

with $k-1 = m+1$

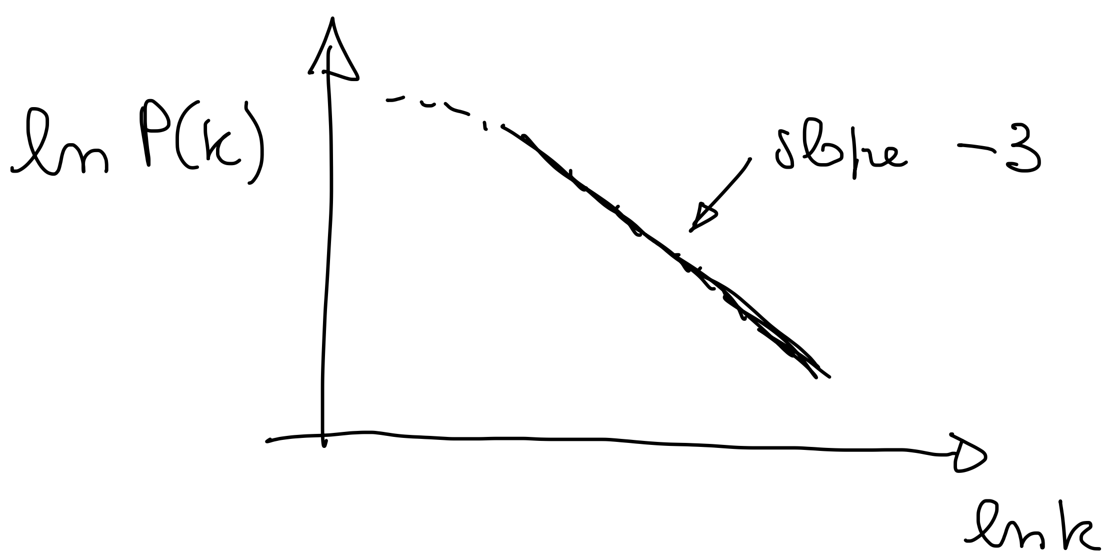
$k = m+2$

$k+2 = m+4$

$$P(m+2) = \frac{m+1}{m+4}\,\frac{m}{m+3} P(m)$$

$$P(k) = \frac{(m+2)(m+1)m}{(k+2)(k+1)k} \cdot \underbrace{P(m)}_{\frac{2}{m+2}} = \frac{(\cancel{m+2})(m+1)m}{(k+2)(k+1)k}\,\frac{2}{\cancel{m+2}}$$

$$\boxed{P(k) = \frac{2m(m+1)}{k(k+1)(k+2)} \quad \text{for } k \geq m}$$



Remember that in the MF approximation we had

$$P(k) \simeq \frac{2m^2}{k^3} \quad \text{(not very different !!)}$$

IMPORTANT: Submit your coursework (PDF)

Assessed coursework 4

closes on Wed 20 March

at 5 PM