Queen Mary University of London

Complex Networks (MTH6142)
Solutions of Formative Assignment 6

- **The moments of a power-law degree distribution**
  Consider a network of N nodes with power-law degree distribution
  $$P(k) = Ck^{-\gamma} \tag{1}$$
  where C is the normalization constant and the power-law exponent γ is greater than one, i.e. $\gamma > 1$.
  Assume that the maximum degree K is given by
  $$K = \min(N, N^{1/(\gamma-1)}) \tag{2}$$
  and the minimum degree is given by $k_{min} = 1$.

  a) Evaluate, in the continuous approximation, the value of the normalization constant C determined by the equation
  $$1 = \int_1^K dkP(k). \tag{3}$$

  b) Evaluate, in the continuous approximation, $\langle k \rangle$ and $\langle k^2 \rangle$.

  c) Perform the limit $N \to \infty$ of $\langle k \rangle$ and $\langle k^2 \rangle$ obtained in part b) for $\gamma \in (1, 2]$.

  d) Perform the limit $N \to \infty$ of $\langle k \rangle$ and $\langle k^2 \rangle$ obtained in part b) for $\gamma \in (2, 3]$.

  e) Perform the limit $N \to \infty$ of $\langle k \rangle$ and $\langle k^2 \rangle$ obtained in part b) for $\gamma \in (3, \infty)$.

  f) When is the network scale-free?

- *Notes on solution*

  a) The normalization constant C is determined , in the continuous approximation, by
  $$1 = \int_1^K dkP(k), \tag{4}$$

which reads

$$1 = C\int_1^K dk k^{-\gamma} = C\frac{1}{1-\gamma}[K^{1-\gamma}-1]. \tag{5}$$

Therefore we have

$$C = \frac{\gamma-1}{1-K^{1-\gamma}}. \tag{6}$$

Using the expression of K as a function of N, valid for $N \gg 1$

$$K = \begin{cases} N & \text{for} \quad \gamma \in (1,2] \\ N^{1/(\gamma-1)} & \text{for} \quad \gamma \in (2,\infty) \end{cases} \tag{7}$$

we obtain

$$C = \begin{cases} \frac{\gamma-1}{1-N^{1-\gamma}} & \text{for } \gamma \in (1,2] \\ \frac{\gamma-1}{1-N^{-1}} & \text{for } \gamma \in (2,\infty) \end{cases} \tag{8}$$

b) In the continuous approximation we can estimate $\langle k^n \rangle$ as in the following

$$\langle k^n \rangle = \int_1^K k^n P(k) dk = \int_1^K C k^{n-\gamma} dk = \begin{cases} C\frac{1}{n+1-\gamma}[K^{n+1-\gamma}-1] & \text{for } \gamma < n+1, \\ C\log K & \text{for } \gamma = n+1, \\ C\frac{1}{\gamma-n-1}[1-K^{n+1-\gamma}] & \text{for } \gamma > n+1. \end{cases}$$

In this expression the cutoff $K = \min(N, N^{1/(\gamma-1)})$ is given by

$$K = \begin{cases} N & \text{for} \quad \gamma \in (1,2] \\ N^{1/(\gamma-1)} & \text{for} \quad \gamma \in (2,\infty) \end{cases} \tag{9}$$

for $N \gg 1$. Therefore $\langle k \rangle = \langle k^n \rangle$ with $n = 1$ is given by

$$\langle k \rangle = \begin{cases} C\frac{1}{2-\gamma}[N^{2-\gamma}-1] & \text{for} \quad \gamma \in (1,2), \\ C\log N & \text{for} \quad \gamma = 2, \\ C\frac{1}{\gamma-2}[1-N^{\frac{2-\gamma}{\gamma-1}}] & \text{for} \quad \gamma \in (2,\infty). \end{cases}$$

Moreover $\langle k^2 \rangle = \langle k^n \rangle$ with $n = 2$ is given by

$$\langle k^2 \rangle = \begin{cases} C\frac{1}{3-\gamma}[N^{3-\gamma}-1] & \text{for } \gamma \in (1,2], \\ C\frac{1}{3-\gamma}[N^{\frac{3-\gamma}{\gamma-1}}-1] & \text{for } \gamma \in (2,3), \\ C\frac{1}{2}\log N & \text{for } \gamma = 3, \\ C\frac{1}{\gamma-3}[1-N^{\frac{3-\gamma}{\gamma-1}}] & \text{for } \gamma \in (3,\infty). \end{cases}$$

c) Assuming $\gamma \in (1,2]$, in the limit $N \to \infty$ we have

$$C \to C^\star = \gamma - 1, \tag{10}$$

and

$$\begin{aligned} \langle k \rangle &\to \infty \\ \langle k^2 \rangle &\to \infty. \end{aligned} \tag{11}$$

d) Assuming $\gamma \in (2, 3]$, in the limit $N \to \infty$ we have

$$C \to C^{\star} = \gamma - 1, \tag{12}$$

and

$$\begin{aligned} \langle k \rangle &\to C^{\star}\frac{1}{\gamma - 2} = \frac{\gamma - 1}{\gamma - 2} \\ \langle k^2 \rangle &\to \infty. \end{aligned} \tag{13}$$

e) Assuming $\gamma > 3$, in the limit $N \to \infty$ we have

$$C \to C^{\star} = \gamma - 1, \tag{14}$$

and

$$\begin{aligned} \langle k \rangle &\to C^{\star}\frac{1}{\gamma - 2} = \frac{\gamma - 1}{\gamma - 2} \\ \langle k^2 \rangle &\to C^{\star}\frac{1}{\gamma - 3} = \frac{\gamma - 1}{\gamma - 3}. \end{aligned} \tag{15}$$

f) The network is scale-free when the average degree $\langle k \rangle$ is finite and the second moment $\langle k^2 \rangle \to \infty$ in the limit $N \to \infty$.
This occurs for $\gamma \in (2, 3]$.