Week 8 Lecture 2

5.5 THE BA MODEL

Barabasi - Albert (Science 1999)

A model to construct SCALE-FREE which reproduces the mechanisms by which real-world networks evolve

WWW, Internet, actor collaborations, citation networks, protein interactions

- GROWTH : # of nodes N and # of links L increase in time

  new pages in WWW, new articles ...

- PREFERENTIAL ATTACHMENT : highly connected nodes are more likely to acquire new links

  popular webpages, popular articles ...

DEF BA MODEL

At time $t=1$ we start with a network of $\begin{cases} n_0 & \text{nodes} \\ m_0 & \text{links} \end{cases}$

At each time $t>1$

① A new node with m new links is added to the network

GROWTH

$$(m \leq n_0)$$

② Every new link is attached to an existing node i with a probability:

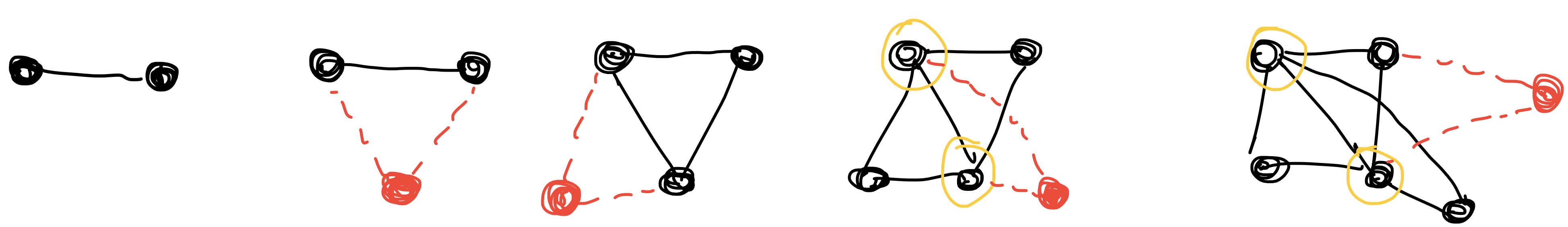
$$\Pi_i = \frac{k_i}{\sum_j k_j}$$

degree of node i

PREFERENTIAL ATTACHMENT

Visualisation in the case $n_0 = 2$ $m_0 = 1$ $m = 2$

$t=1$ $t=2$ $t=3$ $t=4$ $t=5$ $t=\ldots$

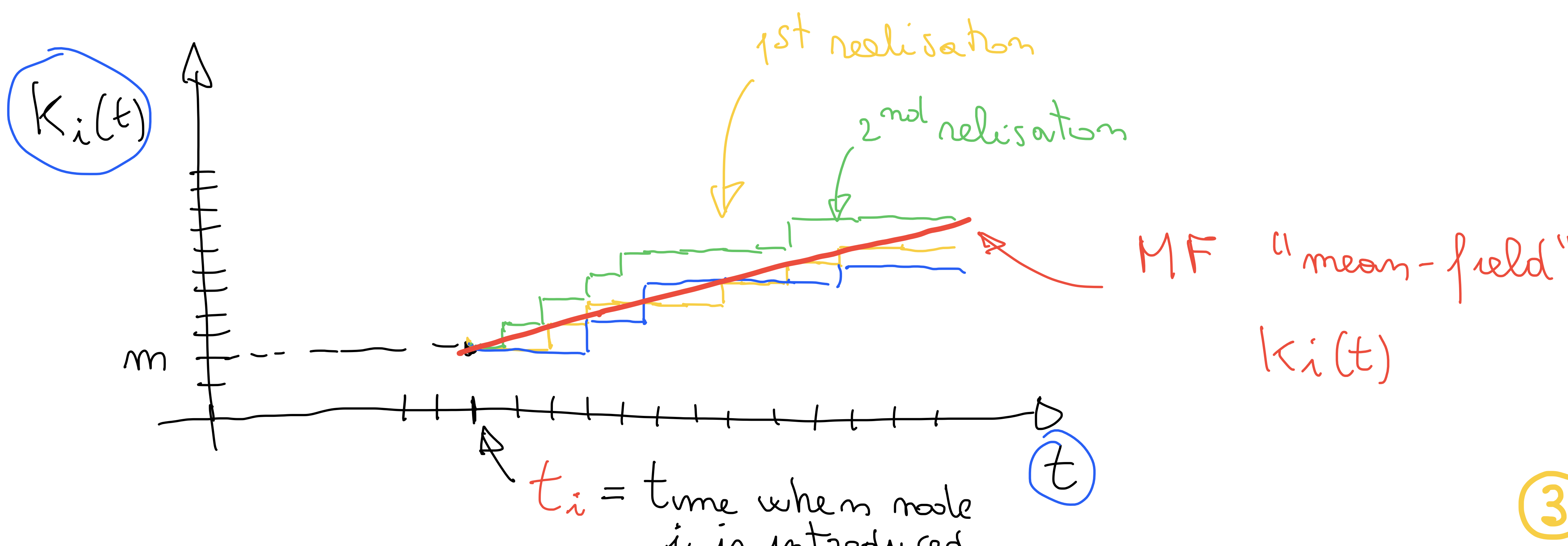


DEF MEAN-FIELD APPROXIMATION

Consists in

- the continuous approximation of the degree $k_i(t)$ of a node i at time t (i.e. both k and t are treated as real numbers
- the stochastic variable $k_i(t)$ is replaced by its average over different realisations of the growth process

deterministic variable



We can write a differential equation for the mean-field $K_i(t)$

$$\begin{cases} \dfrac{dK_i(t)}{dt} = \tilde{\Pi}_i & \text{for } t > t_i \\ K_i(t_i) = m \end{cases}$$

initial condition

time of arrival of node i

with $\tilde{\Pi}_i = m \cdot \Pi_i = m \dfrac{k_i}{\sum_j k_j}$

expected increase in the # of links of node i at time t

probability that a new link connects to node i

MF

Mean-field approximation of the BA model

$$\begin{cases} \dfrac{dK_i}{dt} = m \dfrac{k_i}{\sum_j k_j} & \text{for } t > t_i \\ K_i(t_i) = m \end{cases} \quad (*)$$

PROPOSITION

In the MF approximation the degree $k_i(t)$ of a node i arrived in the network at time t_i is given by

$$k_i(t) = m\left(\frac{t}{t_i}\right)^{\beta} \quad \text{for } t \geq t_i \quad \text{with } \beta = \frac{1}{2}$$

Proof

Observe that $\sum_j k_j = 2\,\underbrace{L(t)}_{\text{total \# of links at time } t}$

$$L(t) = m_0 + m(t-1) \simeq mt \quad \text{for } t \gg 1$$

Therefore for $t \gg 1$ $\quad \sum_j k_j \simeq 2mt$

and the MF equation ⊛ becomes:

$$\frac{dk_i}{dt} = \frac{m k_i}{\sum_j k_j} \simeq \frac{\cancel{m} k_i}{2 \cancel{m} t} = \frac{k_i}{2t} \quad \text{for } t \gg 1$$

$$\frac{dk_i}{k_i} = \frac{dt}{2t}$$ and we can now integrate between t_i and t

$$\int_{k_i(t_i)}^{k_i(t)} \frac{dk_i'}{k_i'} = \frac{1}{2}\int_{t_i}^{t} \frac{1}{t'}\,dt'$$

$$\underbrace{k_i(t_i)}_{m}$$

$$\ln \frac{k_i(t)}{m} = \frac{1}{2} \ln \frac{t}{t_i}$$

$$\frac{k_i(t)}{m} = \left(\frac{t}{t_i}\right)^{\frac{1}{2}}$$

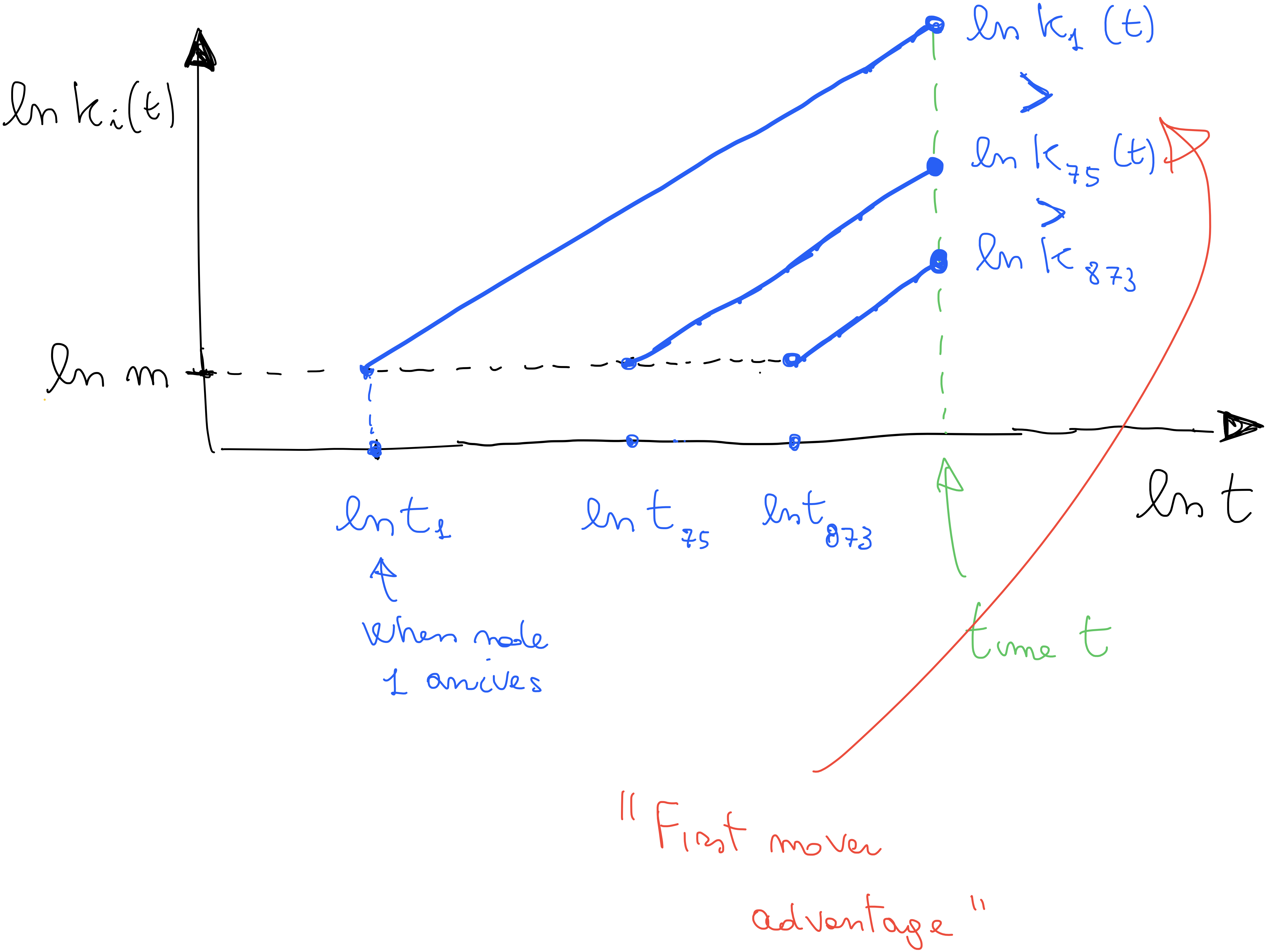
for $t \geq t_i$

which is what we wanted to have

Let's rewrite this

$$\underbrace{\ln k_i(t)}_{Y_i} = \underbrace{\ln m - \frac{1}{2}\ln t_i}_{A_i} + \frac{1}{2}\underbrace{\ln t}_{X}$$

$$Y_i = A_i + \frac{1}{2} X$$



$\ln k_i(t)$
$\ln m$
$\ln k_1(t)$
$>$
$\ln k_{75}(t)$
$>$
$\ln k_{873}$
$\ln t_1$
$\ln t_{75}$
$\ln t_{873}$
$\ln t$
when node 1 arrives
time t
"First mover advantage"