WEEK 8 Lecture 1

CHAPTER 5
SCALE-FREE NETWORKS

5.1 INTRODUCTION

– Real-world networks are SPARSE ≡ "$\langle k \rangle$ is finite and small"

$$L \ll N^2$$

$$\langle k \rangle = \frac{2L}{N} \ll N$$

which is in agreement with POISSON NETWORKS

$$p = \frac{c}{N-1} \qquad c = \langle k \rangle$$

– However real-world networks CANNOT be described by Poisson networks as they have a BROAD DEGREE DISTRIBUTION with a power law tail $P(k) \sim k^{-\gamma}$ for $k \gg 1$

with $\gamma \in (2,3]$

with "large" fluctuations: large values of σ_k ← standard deviation

which is different from a Poisson distribution

$$\langle k \rangle = c$$

$$\sigma_k = \sqrt{c} = \sqrt{\langle k \rangle}$$

5.2 POWER-LAW NETWORKS

DEF POWER-LAW NETWORKS

Networks with a power-law degree distribution

$$P(k) = C\, k^{-\gamma} \quad \text{for } k = k_{min}, \dots, K$$

with $\boxed{\gamma > 1}$ ← POWER-LAW EXPONENT

minimum and maximum degree

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C is a normalisation constant fixed by

$$1 = \sum_{k=k_{min}}^{K} P(k) = C \sum_{k=k_{min}}^{K} k^{-\gamma}$$

$$C = \frac{1}{\sum_{k=k_{min}}^{K} k^{-\gamma}}$$

When $N \to \infty$ $(K \to \infty)$

$$C = \frac{1}{\sum_{k=k_{min}}^{\infty} k^{-\gamma}} = \frac{1}{\zeta(\gamma, k_{min})}$$

if $\gamma > 1$

incomplete RIEMANN ZETA FUNCTION

PROPOSITION

The degree distribution of power-law networks can be plotted as a straight line in a log-log plot

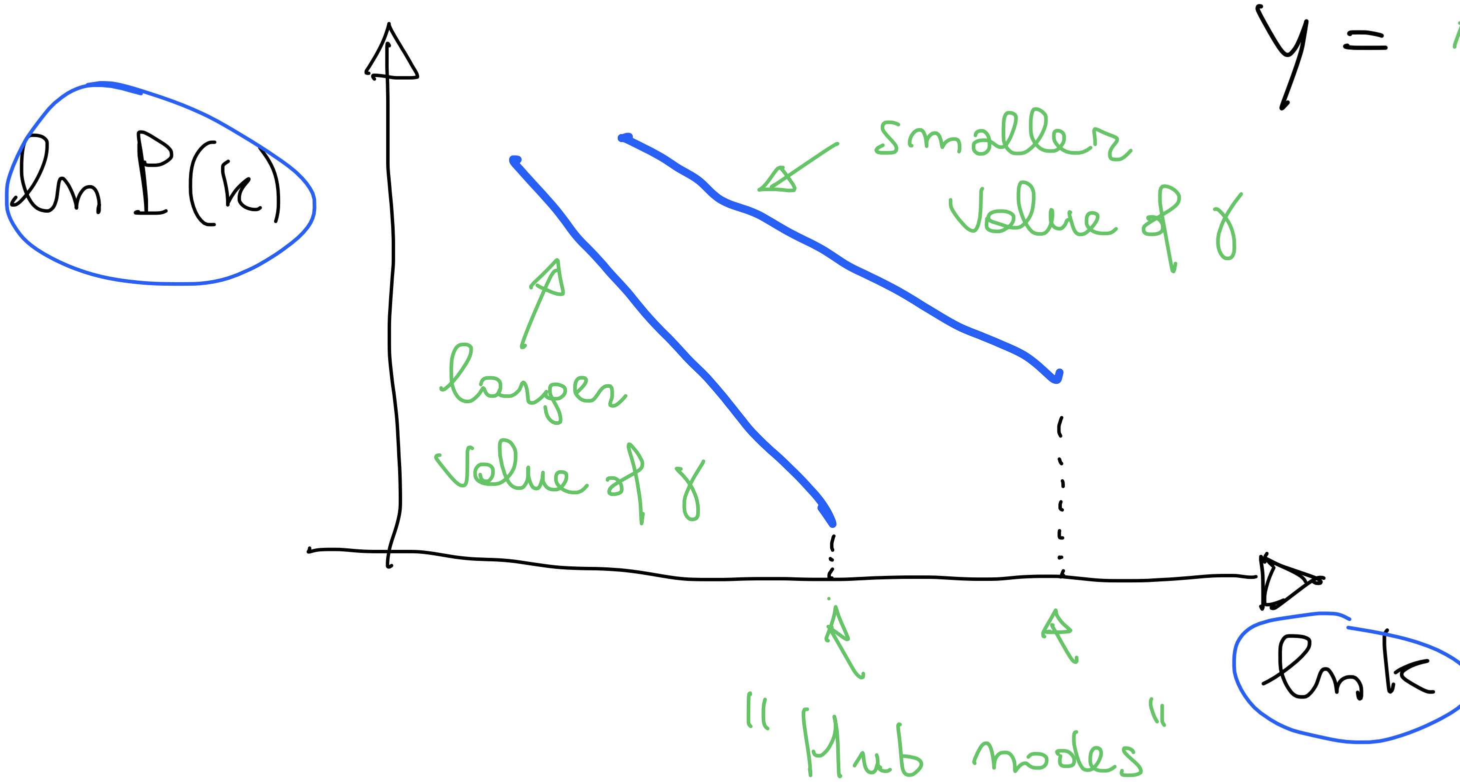
Proof $$P(k) = C \cdot k^{-\gamma}$$

$$\ln P(k) = \underbrace{\ln C}_{A} - \gamma \ln k$$

y axis

x axis

$$y = A - \gamma x$$



In order to characterise

- ✓ normalisation constant C
- moments of the distribution
- ✓ the expected maximum degree ← NATURAL CUTOFF

it will be useful to treat the degree as a <u>CONTINUOUS VARIABLE</u>

DEF CONTINUOUS APPROXIMATION

The node degree k is assumed to be a continuous variable taking real positive values

$$P(k) = C\,k^{-\gamma} \quad \text{with} \quad k \in \mathbb{R}^{+} \qquad k \in [k_{min}, K]$$

$$\gamma > 1 \quad \text{and} \quad C = \left[\int_{k_{min}}^{K} k^{-\gamma}\, dk\right]^{-1}$$

We can easily evaluate the integral

$$C = \left[\frac{k^{-\gamma+1}}{-\gamma+1}\Bigg|_{k_{min}}^{K}\right]^{-1} = \frac{-\gamma+1}{K^{-\gamma+1} - k_{min}^{-\gamma+1}} \xrightarrow[N\to\infty]{} (\gamma-1)\,k_{min}^{\gamma-1}$$

$N \to \infty$ $\to 0$ since $\gamma > 1$

$$C \simeq (\gamma-1)\,k_{min}^{\gamma-1}$$

DEF NATURAL CUTOFF

The natural cutoff of a power-law network with $P(k) = C k^{-\gamma}$ is the expected maximum degree in a network with N nodes

$\bar{K}$ can be smaller than N

PROPOSITION

The natural cutoff $\bar{K}$ of a power-law network with $\gamma > 1$ diverges in the large network limit $(N \to \infty)$ and can be estimated to be

$$\bar{K} \simeq \begin{cases} K_{min} N^{\frac{1}{\gamma - 1}} & \text{for } \gamma > 2 \\ N & \text{for } \gamma \in (1, 2] \end{cases}$$

Proof: $\bar{K}$ is defined by imposing that the expected # of nodes with degree $k > \bar{K}$ is smaller than 1 meaning $\bar{K}$ is extracted from: $N \sum_{k=\bar{K}}^{\infty} P(k) = 1$

$$N C \sum_{k=\bar{K}}^{\infty} k^{-\gamma} = 1$$

continuous approximation

$$C \int_{\bar{K}}^{\infty} k^{-\gamma} \, dx = \frac{1}{N}$$

$$C \left[\frac{k^{-\gamma+1}}{-\gamma+1} \right]_{\bar{K}}^{\infty} = C \left[0 - \frac{\bar{K}^{-\gamma+1}}{-\gamma+1} \right] = \frac{1}{N}$$

$$C \frac{\bar{K}^{-\gamma+1}}{\gamma-1} = N \qquad \cancel{(\gamma-1)} \, k_{min}^{\gamma-1} \frac{\bar{K}^{-\gamma+1}}{\cancel{\gamma-1}} = \frac{1}{N}$$

$$C \simeq (\gamma-1) \, k_{min}^{\gamma-1}$$

$$\left(\frac{\bar{K}}{k_{min}} \right)^{-\gamma+1} = N^{-1}$$

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$$\frac{\bar{K}}{k_{min}} = N^{\frac{1}{\gamma-1}}$$

$$\boxed{\bar{K} \simeq k_{min} N^{\frac{1}{\gamma-1}}}$$

which is valid if $\frac{1}{\gamma-1} < 1$ $\quad \gamma - 1 > 1$

$$\boxed{\gamma > 2}$$

otherwise $\boxed{\bar{K} \simeq N}$

5.3 MOMENTS

PROPOSITION $\langle k^n \rangle$

The n-th moment $\langle k^n \rangle$ of a power-law degree distribution $P(k) = C k^{-\gamma}$ with $k \in [k_{min}, \bar{K}]$ and $\gamma > 1$ is given, in the continuous approximation, by:

$$\langle k^n \rangle = \begin{cases} \frac{C}{n+1-\gamma}\left[\bar{K}^{n+1-\gamma} - k_{min}^{n+1-\gamma}\right] & \text{for } n \neq \gamma - 1 \\ C \ln\left[\frac{\bar{K}}{k_{min}}\right] & \text{for } n = \gamma - 1 \end{cases}$$

Proof

$$\langle k^n \rangle = \int_{k_{min}}^{\bar{K}} k^n P(k)\, dk = C \int_{k_{min}}^{\bar{K}} k^{n-\gamma}\, dk$$

$$\langle k^n \rangle = \begin{cases} \dfrac{C}{n+1-\gamma} \, k^{n+1-\gamma} \Big|_{k_{min}}^{K} = \dfrac{C}{n+1-\gamma} \left[K^{n+1-\gamma} - k_{min}^{n+1-\gamma} \right] & \text{if } n-\gamma \neq -1 \\ C \ln k \Big|_{k_{min}}^{K} = C \ln \dfrac{K}{k_{min}} & \text{if } n-\gamma = -1 \end{cases}$$

$n \neq \gamma - 1$

$n = \gamma - 1$

We can now study how moments of different order scale with the network size N

$\boxed{\text{PROPOSITION}}$ $\langle k^n \rangle$ in the limit $N \to \infty$

In the large network limit $N \to \infty$ $(K \to \infty)$ we have that

$$\langle k^n \rangle \to \infty \quad \text{IFF} \quad n \geq \gamma - 1 \quad (\text{or } \gamma \leq n+1)$$

if and only if

Proof

- For $n+1-\gamma=0$

$$\langle k^n\rangle = C \ln \frac{K}{k_{min}} \xrightarrow[\substack{N\to\infty \\ K\to\infty}]{} +\infty$$

- For $n+1-\gamma<0$

$$\langle k^n\rangle = \frac{C}{n+1-\gamma}\left[K^{\overset{<0}{n+1-\gamma}} - k_{min}^{n+1-\gamma} \right] \xrightarrow[\substack{N\to\infty \\ K\to\infty}]{} \frac{C}{\gamma-(n+1)} k_{min}^{n+1-\gamma}$$

- For $n+1-\gamma>0$

$$\langle k^n\rangle = \frac{C}{n+1-\gamma}\left[\underbrace{K^{\overset{>0}{n+1-\gamma}}}_{K\to\infty \;\to\; +\infty} - k_{min}^{n+1-\gamma} \right] \xrightarrow[\substack{N\to\infty \\ K\to\infty}]{} +\infty$$

Therefore $\langle k^n\rangle \xrightarrow[N\to\infty]{} \infty$ iff $\gamma \le n+1$

For $n=1$ $\quad \langle k\rangle \xrightarrow[N\to\infty]{} \infty$ iff $\gamma \le 2$

For $n=2$ $\quad \langle k^2\rangle \xrightarrow[N\to\infty]{} \infty$ iff $\gamma \le 3$

PROPOSITION

The average degree $\langle k \rangle$ and the second moment $\langle k^2 \rangle$ of a power-law network with power-law exponent $\gamma > 1$ allow to distinguish 3 different regimes

$\boxed{\gamma > 3}$ $\langle k \rangle \to$ finite constant $\langle k^2 \rangle \to$ finite constant

$$\sigma = \sqrt{\langle k^2 \rangle - \langle k \rangle^2} \to \text{finite}$$

$\boxed{2 < \gamma \leq 3}$ $\langle k \rangle \to$ finite $\langle k^2 \rangle \to \infty$

$$\sigma = \sqrt{\langle k^2 \rangle - \langle k \rangle^2} \to \infty$$

SCALE-FREE NETWORKS

Fluctuations around the average degree diverges

$\boxed{1 < \gamma \leq 2}$ $\langle k \rangle \to \infty$ $\langle k^2 \rangle \to \infty$

These are usually called DENSE scale-free networks

5.4 SCALE-FREE NETWORKS

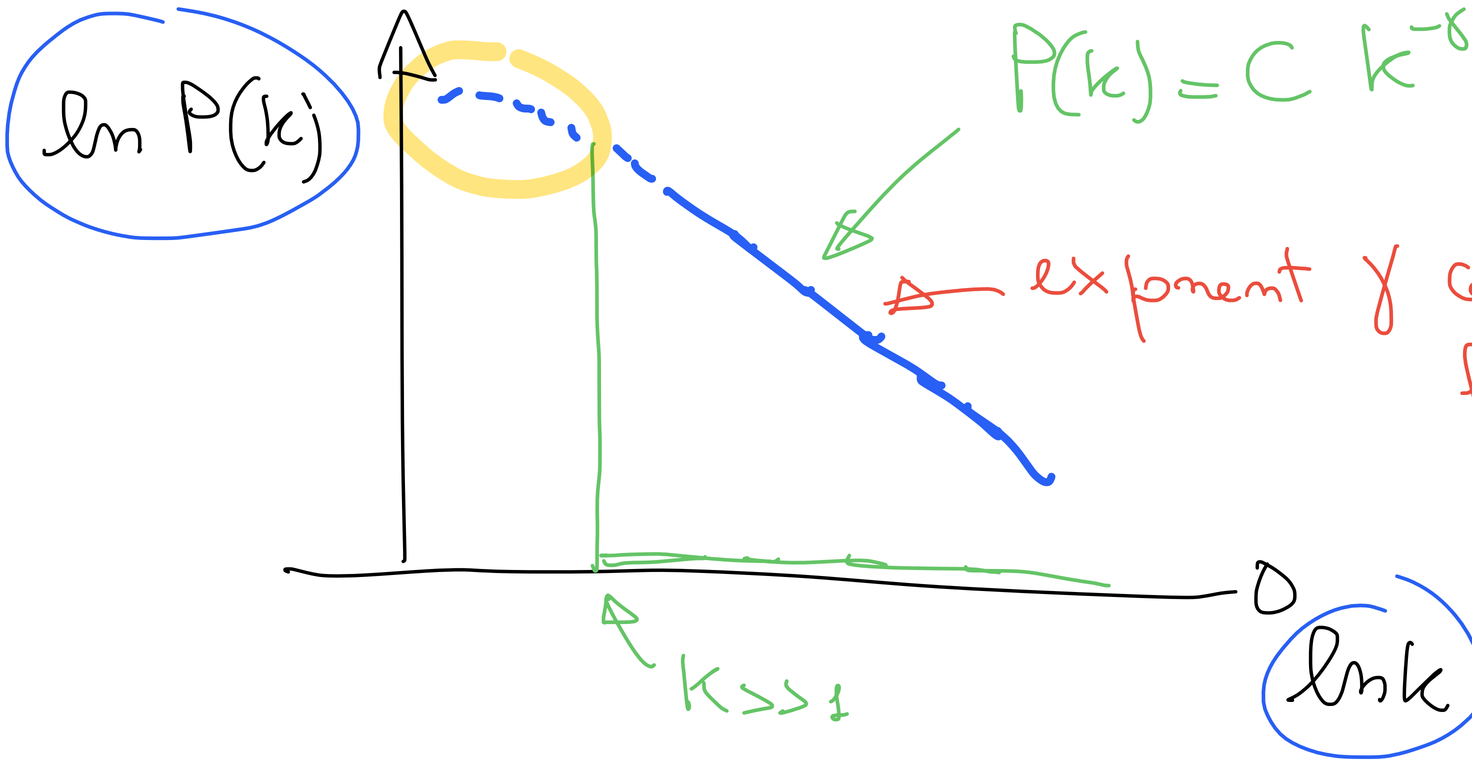
Real-world networks, such as the WWW, social networks, and many biological networks, have very heterogeneous degree distributions

DEF SCALE-FREE NETWORKS

Networks with a degree distribution that, for large values of the degree ($k \gg 1$) can be approximated by a power-law degree distribution:

$$P(k) \simeq C k^{-\gamma} \quad \text{for } k \gg 1$$

with $\gamma \in (2, 3]$



$P(k) = C\, k^{-\gamma}$ $\qquad \gamma \in [2, 3]$

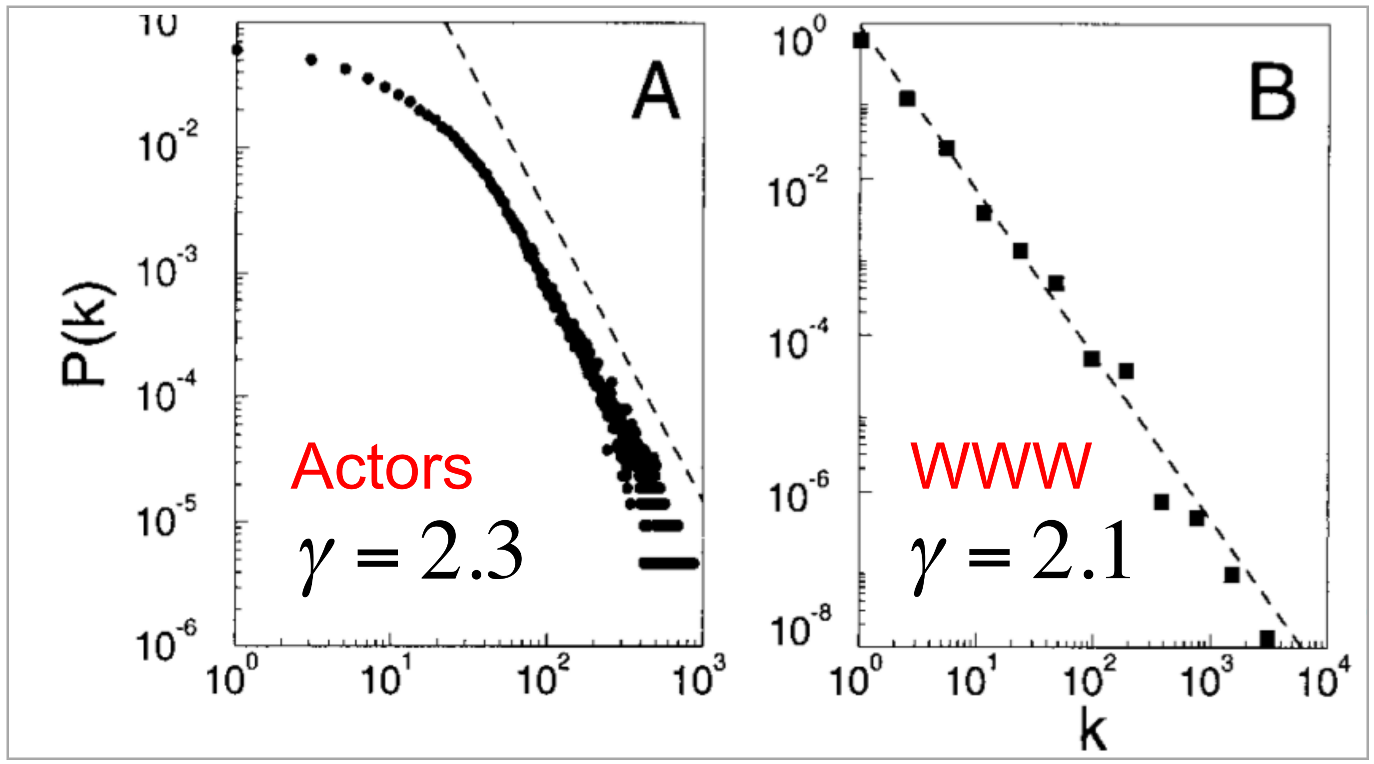
exponent γ can be extracted from the slope

For these networks $\langle k \rangle \longrightarrow$ finite constant

$\langle k^2 \rangle \longrightarrow \infty$

as $N \to \infty$

Filting the values of γ from real-world network data bases



Networks	Exponent	
Internet	2.4	
WWW	2.1	2.7
Protein	2.4	
Metabolic	2.2	2.1
Coauthorship	2.5	
Movie actors	2.3	