WEEK 6 Tutorial

TAKE-HOME MESSAGE from W 5-6 : Random graphs

A $\qquad$ B

$G(N, L) \qquad G(N, p)$

$$L = p \cdot \frac{N(N-1)}{2}$$

$G(N, p)$

BINOMIAL

$$\boxed{P_B(k) = \binom{N-1}{k} p^k (1-p)^{N-1-k} \qquad k = 0, 1, \ldots, N-1}$$

$$\langle k \rangle = (N-1)p$$

$$\langle k(k-1) \rangle = (N-1)(N-2)p^2$$

VARIANCE $\sigma^2 = (N-1)p(1-p)$

Using generating functions

$$G_B(x) = (px + 1 - p)^{N-1}$$

$$p = \frac{c}{N-1}$$

when $N \to \infty$

POISSON NETWORKS

POISSON

$$\boxed{P_p(k) = \frac{c^k}{k!} e^{-c} \qquad k = 0, 1, \ldots}$$

$$\langle k \rangle = c$$

$$\langle k(k-1) \rangle = c^2$$

VARIANCE $\sigma^2 = c$

$$G_p(x) = e^{cx} \cdot e^{-c}$$

TRANSITIONS IN RANDOM GRAPHS

$$\boxed{p = \frac{a}{N^z} = a N^{-z}}$$

$\boxed{\text{For } z > 1}$

$$\langle k \rangle = p(N-1) \xrightarrow[N \to \infty]{} 0$$

NO GIANT COMPONENT (because $\langle k \rangle < 1$)

$$\langle N^{\text{Triangles}} \rangle \xrightarrow[N \to \infty]{} 0$$

(because $z_c^{\text{triangles}} = 1$)

$\boxed{\text{For } z < 1}$

$\langle k \rangle = p(N-1) \xrightarrow[N \to \infty]{} \infty$ (because $\langle k \rangle > 1$)

YES GIANT COMPONENT

$\langle N^{\text{Triangles}} \rangle \longrightarrow \infty$ (because $z_c^{\text{triangles}} = 1$)

We can have 4-cliques

but only when $z \leq \frac{2}{3}$ (because $z_c^{\text{4-cliques}} = \frac{2}{3}$)

$\boxed{\text{For } z = 1}$

$\langle k \rangle = p(N-1) \xrightarrow[N \to \infty]{} a = c$

POISSON NETWORKS with $\langle k \rangle = c = a$

$\boxed{\text{For } c < 1}$	$\boxed{\text{For } c = 1}$	$\boxed{\text{For } c > 1}$
SUBCRITICAL	CRITICAL	SUPERCRITICAL
NO GIANT		YES GIANT

$\langle N^{\text{triangles}} \rangle \xrightarrow[N \to \infty]{} \frac{c^3}{6}$

This is a finite #

So the density of triangles is infinitesimal

From FA 4

2

- **2. A given random network**
Consider a random network in the ensemble $\mathbb{G}(N,p)$ with $N = 4 \times 10^6$ nodes and a linking probability $p = 10^{-4}$.
  - (a) Calculate the average degree $\langle k \rangle$ of this network.
  - (b) Calculate the standard deviation σ_P using the approximated degree distribution given by Eq. (2). Poisson distribution
  - (c) Assume that you observe a node with degree 2×10^3. How many standard deviations is this observation from the mean? Is this an expected observation or is this an unexpected observation?

$$\mathbb{G}(N,p) \qquad N = 4 \cdot 10^6 \qquad p = 10^{-4}$$

a

$$\langle k \rangle = p(N-1) = 10^{-4}(4 \cdot 10^6 - \cancel{1}) \simeq 400$$

b

$P(k)$ approximated by a Poisson with $c = \langle k \rangle = 400$

$$P(k) = \frac{c^k}{k!} e^{-c} = \frac{400^k}{k!} e^{-400}$$

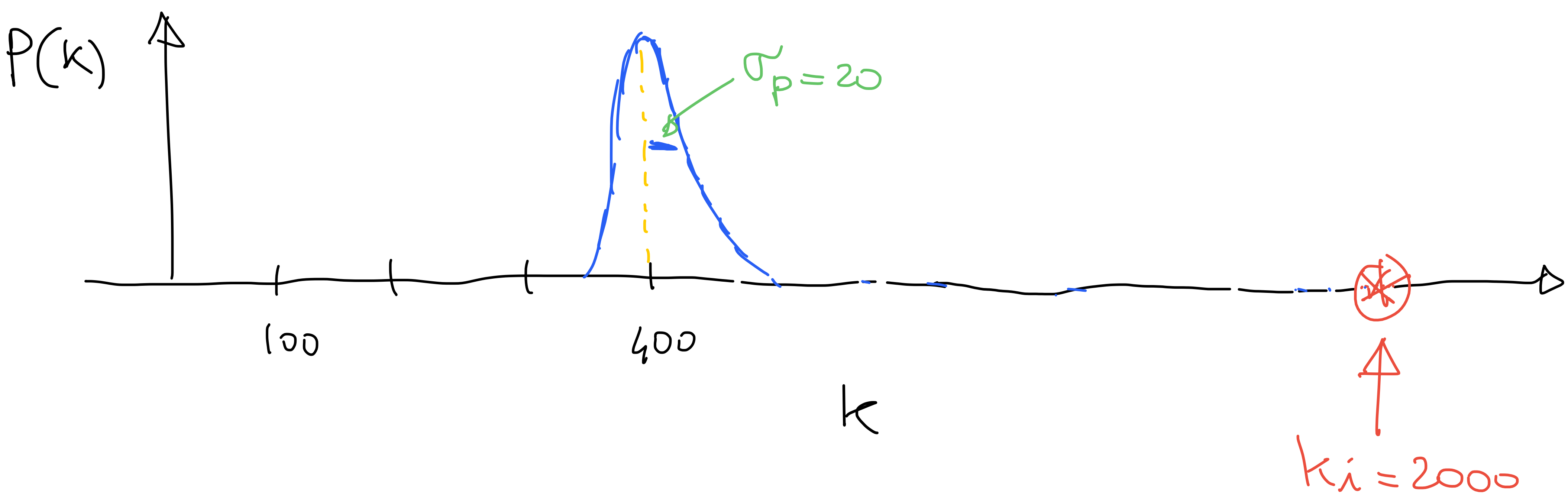
Variance $\sigma_P^2 = c$ Standard deviation $\sigma_P = \sqrt{c} = \sqrt{400} = 20$

4

c

$k_i = k = 2 \cdot 10^3$

$$\frac{k - \langle k \rangle}{\sigma_p} = \frac{2 \cdot 10^3 - 400}{20} = \underline{80}$$



Very very unexpected

From FA 5

- **1. Random networks in the $\mathbb{G}(N,p)$ ensemble**

Assume that $p = a/N^z$, where $a > 0$ and $z \geq 0$, and a, z independent of N.

(a) Determine the average degree $\langle k \rangle$ in the limit $N \to \infty$ for the following values of the parameters

(i) $a = 0.5, z = 1$;
(ii) $a = 2, z = 1$;
(iii) $a > 0,\ z = 2$;
(iv) $a > 0, z = 0.5$.

(b) In which of the above cases does the random network contain a giant component in the limit $N \to \infty$?.

(c) Given $p = a/N^z$ with generic values of $a > 0, z \geq 0$ determine the average degree $\langle k \rangle$ in the large network limit $N \to \infty$.

(d) Determine the conditions on a and z for these random networks to be subcritical, i.e. with a fraction S of nodes in the giant component given by $S = 0$ in the $N \to \infty$ limit.

(e) Determine the conditions on a and z for these random networks to be supercritical, i.e. with a non vanishing fraction S of nodes in the giant component ($S > 0$) in the $N \to \infty$ limit.

(f) Determine the conditions on a and z for which these random networks are critical, in the large network limit, i.e. in the limit $N \to \infty$.

$\boxed{1}$

$\mathbb{G}(N,p) \qquad p = \frac{a}{N^z} \qquad a>0 \qquad z \geq 0$

$\frac{a}{b}$

$\langle k \rangle = p(N-1) = \frac{a}{N^z}(N-1)$

GIANT COMPONENT

iff

$\lim_{N\to\infty} \langle k \rangle > 1$

$a = 0.5$, $z = 1$: $\langle k \rangle = \frac{0.5}{N^1}(N-1)$ $\quad \lim_{N\to\infty} \langle k \rangle = \lim_{N\to\infty} 0.5 \frac{N-1}{N} = 0.5 < 1$

NO GIANT COMPONENT

$a = 2$, $z = 1$: $\langle k \rangle = \frac{2}{N^1}(N-1) \xrightarrow[N\to\infty]{} 2 > 1$ YES GIANT

$a > 0$, $z = 2$: $\langle k \rangle = \frac{a}{N^2}(N-1) \xrightarrow[N\to\infty]{} 0 < 1$ NO GIANT

$a > 0$, $z = 0.5$: $\langle k \rangle = \frac{a}{N^{\frac{1}{2}}}(N-1) \xrightarrow[N\to\infty]{} \infty > 1$ YES GIANT

c) $\langle k \rangle = \frac{a}{N^z}(N-\cancel{1}) = \frac{a}{N^{z-1}}$

$$\boxed{\lim_{N\to\infty} \langle k \rangle = \begin{cases} 0 & \text{if } z > 1 \\ a & \text{if } z = 1 \\ \infty & \text{if } z < 1 \end{cases}}$$

d) Subcritical if

$\lim_{N\to\infty} \langle k \rangle < 1 \longrightarrow$ if $z > 1$ or if $\begin{cases} z = 1 \\ a < 1 \end{cases}$

(e) Supercritical if

$$\lim_{N\to\infty} \langle k \rangle > 1 \quad \to \quad \text{if } z<1 \quad \text{or if} \begin{cases} z=1 \\ a>1 \end{cases}$$

(f) Critical

$$\lim_{N\to\infty} \langle k \rangle = 1 \quad \to \quad \text{if} \begin{cases} z=1 \\ a=1 \end{cases}$$

$\boxed{2}$

- **2. Random networks in the $\mathbb{G}(N, p)$ ensemble with $p = c/(N-1)$ where $c > 0$.** Poisson

(a) Calculate the average number of triangles $\mathcal{N}_3^{\text{triangles}}$ in the network, by evaluating first the number of ways to select 3 nodes out of N nodes, and secondly the probability that the selected nodes are all connected to each other.

(b) Show that in the limit $N \to \infty$ the average number of triangles in the network is

$$\mathcal{N}_3^{\text{triangles}} \simeq \frac{1}{6}c^3. \tag{12}$$

This means that the number of triangles is constant, neither growing or vanishing, in the limit of large N.

a

$$\langle N^{\text{triangles}} \rangle = \binom{N}{3} \cdot p^3$$

\# of ways of selecting 3 nodes out of 10

b

$$\lim_{N\to\infty} \langle N^{\text{triangles}} \rangle = \lim_{N\to\infty} \binom{N}{3} p^3 =$$

$$= \lim_{N\to\infty} \frac{N!}{3!\,(N-3)!} \frac{c^3}{(N-1)^3} = \lim_{N\to\infty} \frac{c^3}{3!} \frac{\cancel{N}\,\cancel{(N-1)}\,\cancel{(N-2)}}{\cancel{(N-1)^3}} = \frac{c^3}{3!} =$$

$$= \frac{c^3}{6}$$

$\underbrace{\text{No Lectures in W 7}}$

but do NOT forget to work at QUIZ 3

and submit it by WEDNESDAY