WEEK 6 Lecture 2

We will now concentrate on the expected # of subgraphs in ensembles of random graphs

4.7 NUMBER OF CLIQUES

DEF CLIQUE ← From Sect 2.6

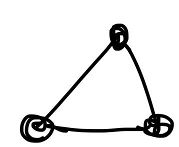
A clique $\mathbb{K}_n$ is a subgraph $G'(V',E')$ with $|V'|=n$ of an undirected network, such that every node is linked to every other node

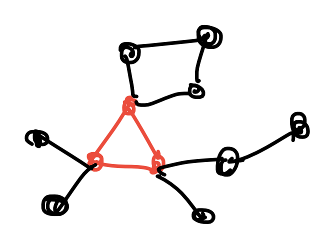
n is the clique SIZE, $n \geq 3$

# of links $= \binom{n}{2} = \frac{n(n-1)}{2}$

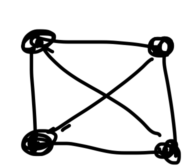
Ex

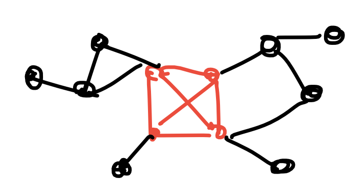
$n=3$ Triangle $\mathbb{K}_3$





$n=4$ 4-clique $\mathbb{K}_4$





PROPOSITION n-CLIQUES

The expected # of n-cliques $\langle N_n^{cliques} \rangle$ in a network of $G(N,p)$ ensemble is:

$$\boxed{\langle N_n^{cliques} \rangle = \binom{N}{n} p^{\frac{n(n-1)}{2}}}$$

Proof

$\binom{N}{n}$ is the # of ways in which we can select n nodes out of the N nodes in the network

$p^{\binom{n}{2}} = p^{\frac{n(n-1)}{2}}$ is the probability of observing $\binom{n}{2}$ links among the n selected nodes

EX For triangles $n=3$

POISSON NETWORK $p=\frac{c}{N-1}$

$$\langle N^{\text{triangles}}\rangle = \binom{N}{3}p^3 = \frac{N!}{3!(N-3)!}p^3 = \frac{N\cancel{(N-1)}\cancel{(N-2)}}{3!}\frac{c^3}{\cancel{(N-1)^3}}$$

large N

In a Poisson network the # of triangles is $\frac{c^3}{3!}$

Hence it is a finite number even if $N\to\infty$

TRIANGLES can be neglected

We will now consider the $G(N,p)$ ensemble with

$$\boxed{p=\frac{a}{N^z}}$$

with $a>0$ and $\boxed{z\geq 0}$

generalised the case $z=1$

PROPOSITION

In the $G(N,p)$ ensemble with $p=\frac{a}{N^z}$ the expected

# of n-cliques in the limit $N\to\infty$ is:

$$\lim_{N\to\infty} \langle N_n^{\text{cliques}} \rangle = \begin{cases} 0 & \text{if } z > \frac{2}{n-1} \\ \frac{1}{n!}\, a^{\frac{n(n-1)}{2}} & \text{if } \boxed{z = \frac{2}{n-1}} \\ \infty & \text{if } z < \frac{2}{n-1} \end{cases}$$

$$\boxed{z_{\text{critical}} = \frac{2}{n-1}}$$

Proof

$$\langle N_n^{\text{cliques}} \rangle = \binom{N}{n} p^{\frac{n(n-1)}{2}} = \frac{N!}{n!\,(N-n)!} \left(\frac{a}{N^z}\right)^{\frac{n(n-1)}{2}} =$$

$N>>1 \longrightarrow$

$$= \frac{\overbrace{N(N-1)(N-2)\cdots(N-n+1)}^{\simeq N^n}}{n!} \; \frac{a^{\frac{n(n-1)}{2}}}{N^{\frac{z\,n(n-1)}{2}}} =$$

$$= \frac{a^{\binom{n}{2}}}{n!} \; N^{\,n - \frac{z\,n(n-1)}{2}} =$$

$$= \frac{a^{\binom{n}{2}}}{n!} \; N^{n\left[\underbrace{1 - \frac{z(n-1)}{2}}_{\tau}\right]}$$

$$\tau < 0 \Rightarrow 1 - \frac{z(n-1)}{2} < 0 \qquad z > \frac{2}{n-1}$$

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Finally

$$\lim_{n\to\infty} \langle N_n^{\text{cliques}} \rangle = \begin{cases} 0 & \text{if } z > \frac{2}{n-1} \\ \frac{a^{\frac{n(n-1)}{2}}}{n!} & \text{if } z = z_c = \frac{2}{n-1} \\ \infty & \text{if } z < \frac{2}{n-1} \end{cases}$$

Ex Triangles

$n=3$ $\quad z_c = \frac{2}{n-1} = \frac{2}{3-1} = 1$ $\quad z_c^{\text{Triangles}} = 1$

$$\lim_{N\to\infty} \langle N^{\text{triangles}} \rangle = \begin{cases} 0 & \text{if } z > 1 \\ \frac{a^{\binom{3}{2}}}{3!} & \text{if } \boxed{z = z_c = 1} \\ \infty & \text{if } z < 1 \end{cases}$$

For $z=1$ $\quad p = \frac{a}{N^z} = \frac{a}{N}$ ← means that $a = c = \langle k \rangle$

↑ POISSON NETWORK

$$\lim_{N\to\infty} \langle N^{\text{triangles}} \rangle = \frac{a^{\frac{3\cdot\cancel{2}}{\cancel{2}}}}{3!} = \frac{c^3}{3!}$$

← coincides with our previous result

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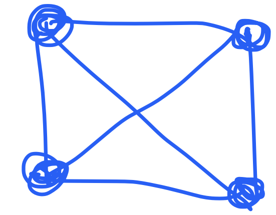
Ex 4-cliques $\mathbb{K}_4$



$$m=4 \qquad z_c = \frac{2}{m-1} = \frac{2}{4-1} = \frac{2}{3} \qquad\qquad z_c^{\mathbb{K}_4} = \frac{2}{3}$$

$$\lim_{N\to\infty} \langle N_4^{\text{cliques}} \rangle = \begin{cases} 0 & \text{if } z > \frac{2}{3} \\ \frac{a^6}{4!} & \text{if } \boxed{z = z_c = \frac{2}{3}} \\ \infty & \text{if } z < \frac{2}{3} \end{cases}$$

In Poisson networks (for which $z=1$) there are NO 4-cliques

To have 4-cliques we require $z \le \frac{2}{3}$ which corresponds to $\langle k \rangle = p(N-1) = \frac{a}{N^z}(N-1) \simeq a\,N^{1-z}$

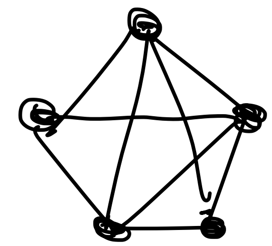
Hence we need at least

$$\langle k \rangle = a\,N^{1-\frac{2}{3}} = a\,N^{\frac{1}{3}} \text{ to have 4-cliques}$$

diverging average degree

Ex

$n = 5$



$$z_c^{K_5} = \frac{1}{2}$$

$$z_c = \frac{2}{n-1} = \frac{2}{5-1} = \frac{1}{2}$$