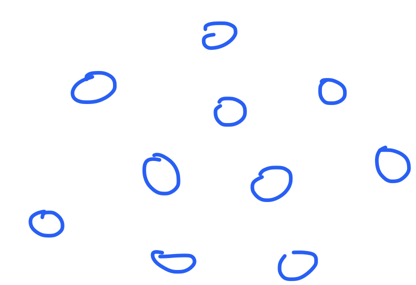
WEEK 6 Lecture 1

4.6 GIANT COMPONENT

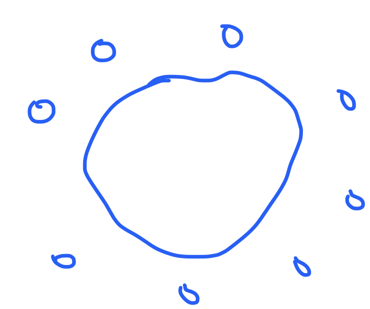
As we increase the # of links in a random network we observe a PHASE TRANSITION in its structural properties

sudden change

$\langle k \rangle < 1$ many small components



$\langle k \rangle > 1$ the largest component becomes "big"



DEF GIANT COMPONENT

A network $G=(V,E)$ has a giant component if its largest (connected) component $H = (V', E')$ is such that

$$|V'| \sim O(N)$$

Big O

In this case $H = (V', E')$ is called the GIANT COMPONENT

$$\lim_{N \to \infty} \frac{|V'|}{N} = \text{constant} > 0$$

the largest component includes a FINITE FRACTION of the nodes

PROPOSITION

A random network in the $G(N, p)$ ensemble with average degree $\langle k \rangle \to c$ for $N \to \infty$, in the limit $N \to \infty$ contains in the giant component a fraction S of the nodes, with S satisfying:

$$\boxed{S = 1 - e^{-cS}} \quad \circledast$$

Proof

$$S = \begin{cases} \text{fraction of nodes in the giant component} \\ \text{probability that a randomly chosen node is in the giant component} \end{cases}$$

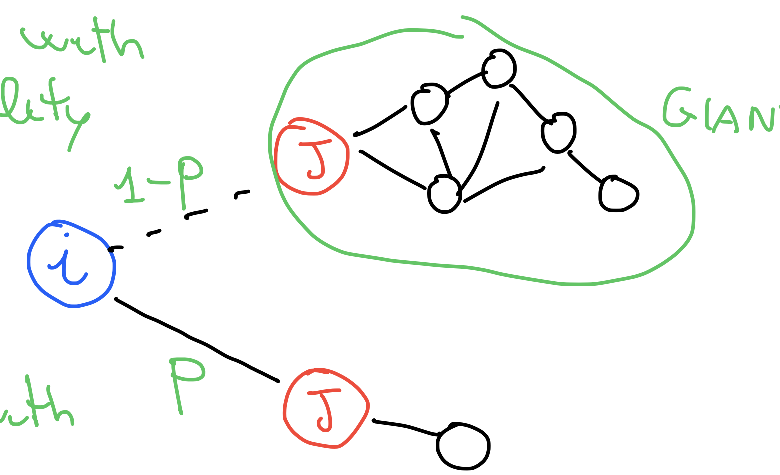
$1 - S$ = probability that a node i is NOT in the GIANT COMPONENT

A node i is NOT in the giant component if every node $J \neq i$

either

(a) i is NOT linked to J ← occurs with probability $1-p$

(b) i is LINKED to J but J is NOT in the giant component ← occurs with probability $p(1-S)$



Hence the probability that either (a) or (b) occurs for a given J is:

$$\cancel{1}-\cancel{p} + p(\cancel{1}-S) = 1 - pS$$

Finally $1 - S = \underbrace{[1-pS][1-pS]\ldots[1-pS]}_{N-1 \text{ terms}}$

Hence we have the following expression for $1-S$ ← the probability that i is NOT in the GIANT COMPONENT

$$(**)\quad \boxed{1-S=(1-pS)^{N-1}} = \left(1-\frac{c}{N-1}S\right)^{N-1}$$

$\langle k \rangle \to c$ by hypothesis
$\langle k \rangle = p(N-1)$

$c = p(N-1)$

$p = \dfrac{c}{N-1}$

a Poisson network when $N \to \infty$

Recall $\quad e^x = \lim_{n\to\infty}\left(1+\dfrac{x}{n}\right)^n$

$n = N-1$
$x = -cS$

Hence $**$ becomes

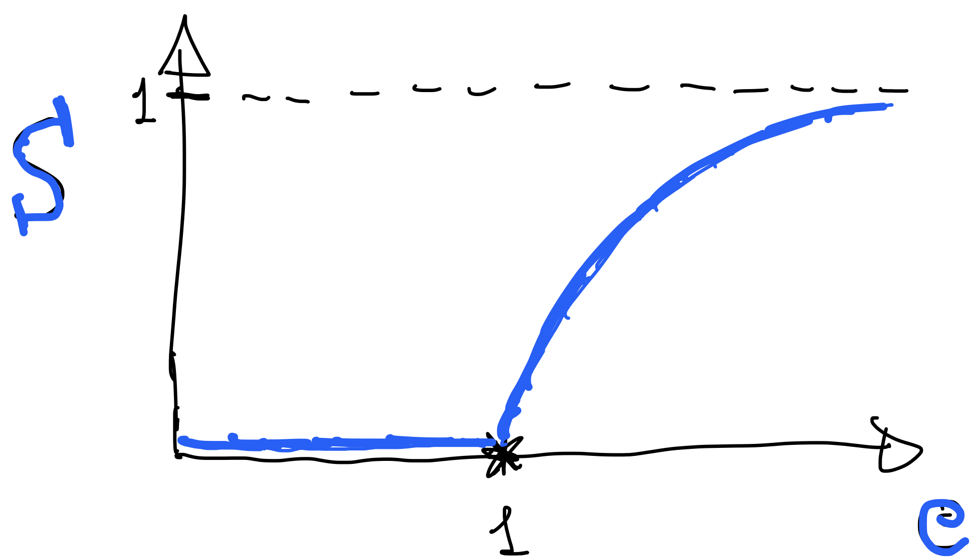
$$1-S = \lim_{N\to\infty}\left(1-\frac{cS}{N-1}\right)^{N-1} = e^{-cS}$$

Finally $\quad \boxed{S = 1-e^{-cS}}$ which is $(*)$

The equation (*) has no closed-form solution

PROPOSITION GIANT COMPONENT

A Poisson network with average degree $\langle k \rangle = c$ has a GIANT COMPONENT iff $\boxed{c > 1}$



$S > 0$ iff $c > 1$

(*) $S = 1 - e^{-cS}$

fraction of nodes in the giant component

→ Eq (*) has always the solution $S = 0$

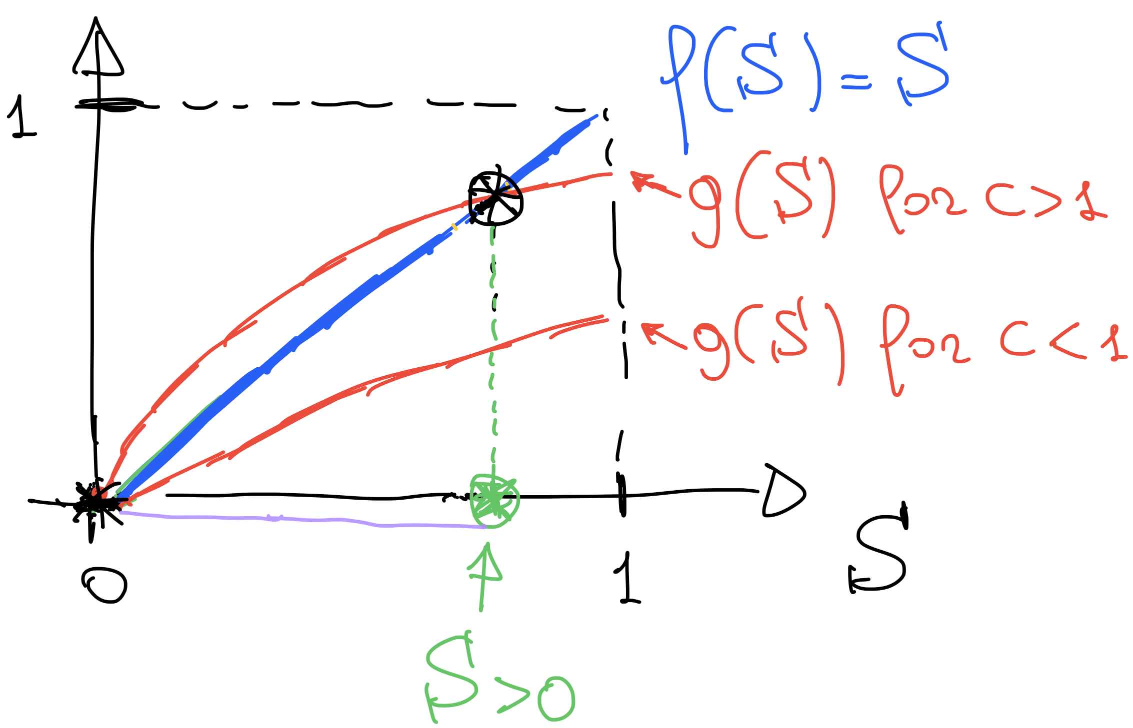
$0 = 1 - e^{-c \cdot 0} = 1 - 1 = 0$

→ Eq (*) can also have a NON-ZERO solution $S > 0$ when $c > 1$

We can find the NON-ZERO solution graphically as the intersection of the two functions

$$\begin{cases} f(S) = S \\ g(S) = 1 - e^{-cS} \end{cases} \qquad \circledast \Leftrightarrow f(S) = g(S)$$

$S \in [0, 1]$ In this range $\begin{cases} 0 \le f(S) \le 1 \\ 0 \le g(S) \le \underbrace{1 - e^{-c}}_{g(1)} < 1 \end{cases}$



$$g(S) = 1 - e^{-cS}$$

$$g'(S) = \underbrace{c}_{c>0}\, e^{-cS} > 0$$

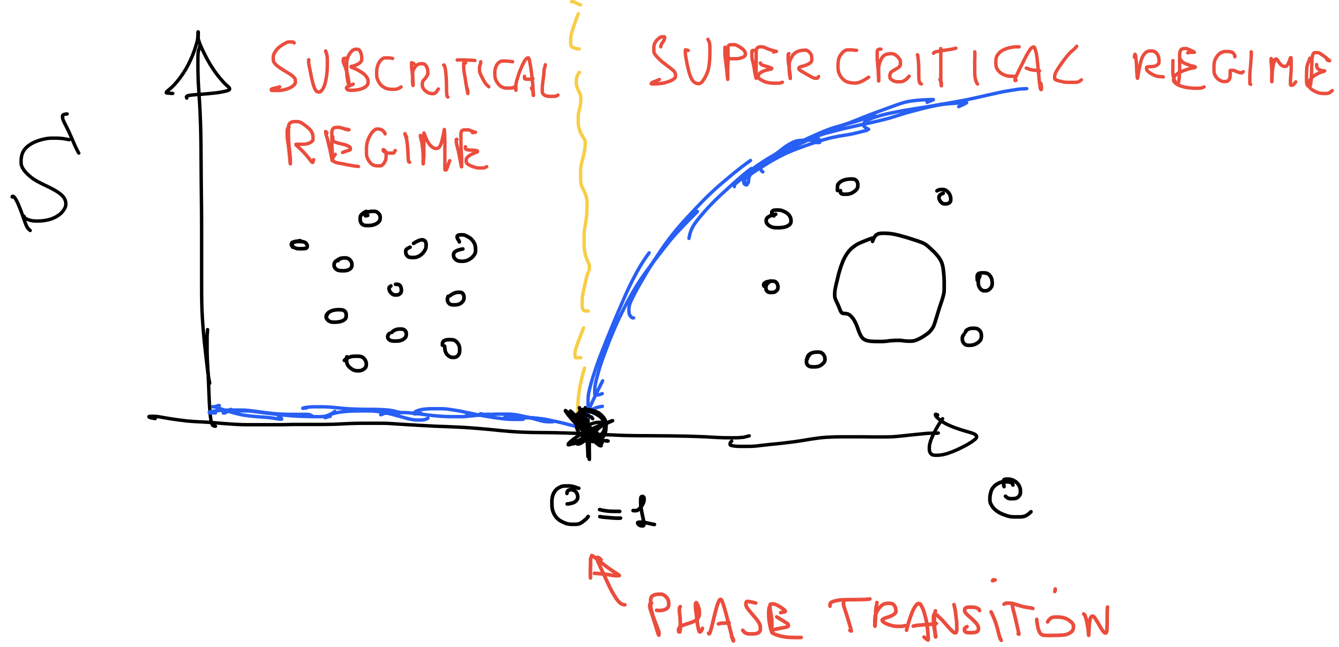
$g(S)$ is an increasing function with max slope at $S = 0$

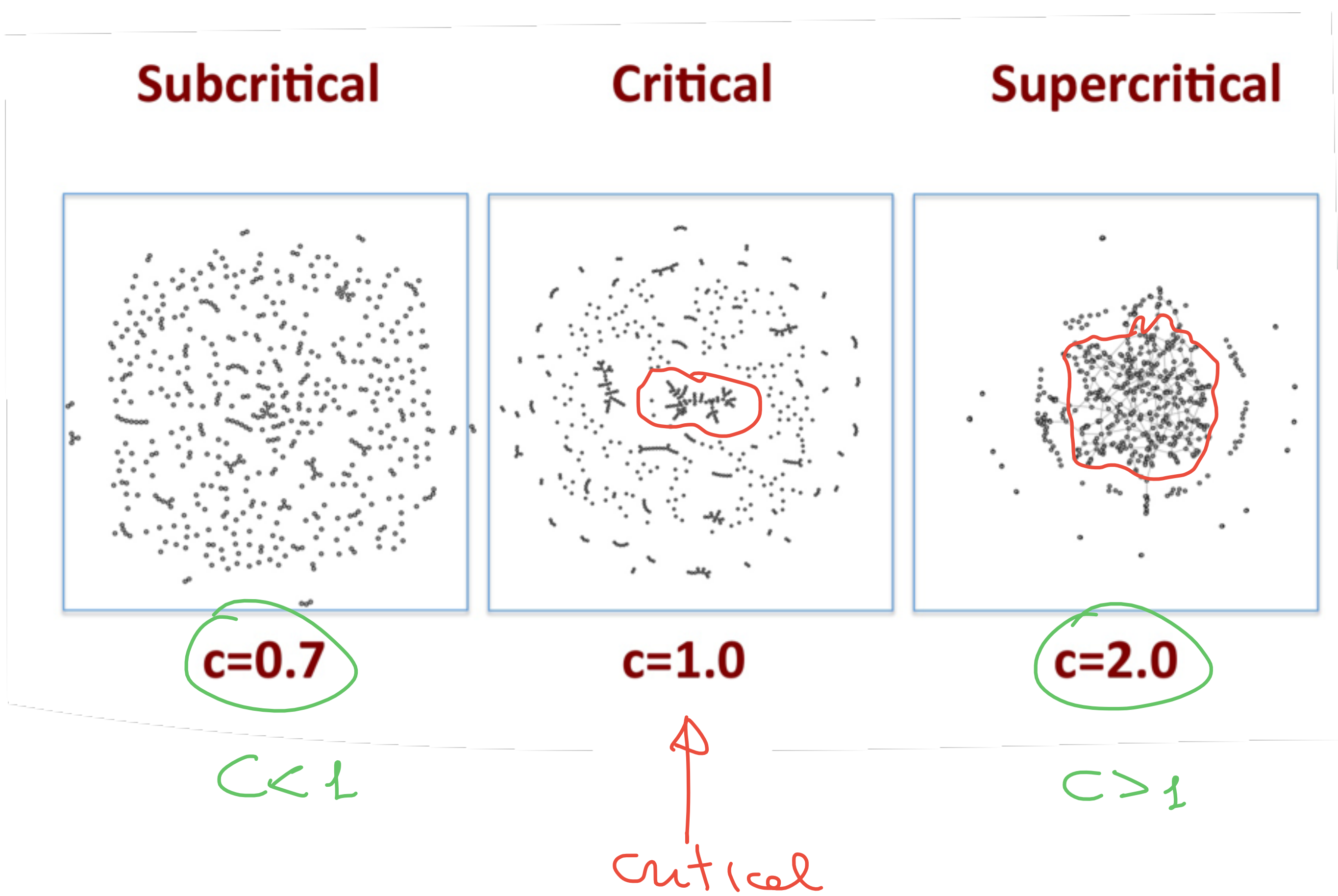
$$g'(0) = c$$

Eq $\ast$ has always $S = 0$ solution

Eq $\ast$ has also solution $S > 0$ when

$c > 1$ ← therefore the network has a GIANT COMPONENT when $c > 1$





Poisson network with $N=500$ nodes

These results extend to random graphs in the $G(N,p)$ ensemble

If $\boxed{\lim_{N\to\infty} \langle k \rangle > 1}$ the network is in the SUPERCRITICAL REGIME $S > 0$

If $\boxed{\lim_{N\to\infty} \langle k \rangle < 1}$ SUBCRITICAL REGIME $S = 0$

If $\boxed{\lim_{N\to\infty} \langle k \rangle = 1}$ CRITICAL POINT $S = 0$

Ex

Consider the $G(N,p)$ ensemble with

$p \equiv p(N)$

a) $p = \dfrac{10}{N^5}$ b) $p = \dfrac{0.5}{N^{\frac{1}{3}}}$ c) $p = \dfrac{1}{N-35}$

Find if we are in the supercritical, subcritical or critical regime

a) $\langle k \rangle = p(N-1) = \dfrac{10}{N^5}(N-1)$

$$\lim_{N\to\infty} \langle k \rangle = \lim_{N\to\infty} 10 \frac{N-1}{N^5} = 0 < 1 \Rightarrow$$ SUBCRITICAL

$S = 0$ No giant component

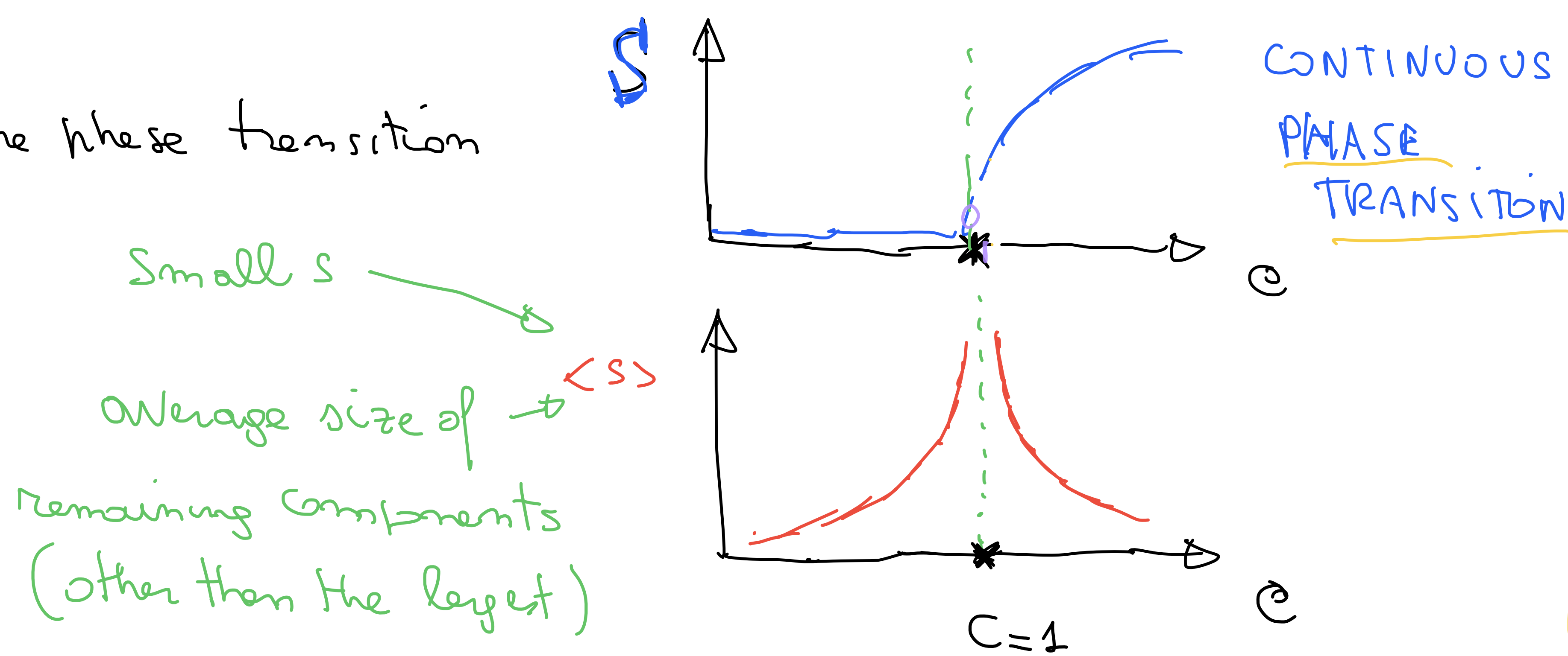
(b) $$\langle k \rangle = p(N-1) = \frac{0.5}{N^{\frac{1}{3}}}(N-1) \xrightarrow[N\to\infty]{} \infty > 1 \Rightarrow$$ SUPERCRITICAL

yes giant component

(c) $$\langle k \rangle = p(N-1) = \frac{1}{N-35}(N-1)$$

$$\lim_{N\to\infty} \langle k \rangle = \lim_{N\to\infty} \frac{N-1}{N-35} = 1 \Rightarrow$$ CRITICAL

More on the phase transition



9

PROPOSITION Critical exponents

For $c = 1 + \varepsilon$ with $0 < \varepsilon \ll 1$ the fraction S of nodes in the giant component increases as

$$S \simeq (c-1)^{\beta} \quad \text{with } \beta = 1$$

CRITICAL EXPONENT

Proof

$$(\ast) \qquad S = 1 - e^{-cS}$$

Since $c = 1 + \varepsilon$ $\quad 0 < \varepsilon \ll 1$
in this region we have $\quad 0 < S \ll 1$

Recalling $\quad e^{-x} = 1 - x + \frac{x^2}{2} - \dots \quad$ for $x \ll 1$

we can expand the exponential in $(\ast)$ with $x = cS \ll 1$

We obtain

$$S = \cancel{1} - \left[\cancel{1} - cS + \frac{1}{2} c^2 S^2 - \dots\right]$$

$$S = cS - \frac{1}{2} c^2 S^2$$

we can divide by $S > 0$

$$1 = c - \frac{1}{2} c^2 S$$

$\beta = 1$

$$S = \frac{2}{c^2}(c-1) = \frac{2}{c} - \frac{2}{c^2} \simeq \underset{f(1)}{0} + \underset{f'(1)}{2}(c-1) = 2(c-1)^{\beta}$$

$c = 1+\varepsilon$ so we can expand around $c=1$

$f(c) = \frac{2}{c} - \frac{2}{c^2}$ $\qquad f(1) = 2-2 = 0$

$f'(c) = -2c^{-2} + 4c^{-3}$ $\qquad f'(1) = -2+4 = 2$

New question: How many links do we have to add to a random network for it to be made by a single connected component?

PROPOSITION **SINGLE CONNECTED COMPONENT**

For $\boxed{\langle k \rangle \simeq \ln N}$ the networks in the $G(N,p)$ ensemble contain a single connected component

Proof

(**) $$1-S = (1-pS)^{N-1}$$

$$\underbrace{(1-S)N}_{\text{\# of nodes NOT in the giant component}}$$

$$(1-S)N \simeq 1 \leftarrow \text{threshold for appearance of a single connected component}$$

$$\Downarrow$$

$$S \simeq 1 - \frac{1}{N}$$

The Eq. (**) becomes

$$\cancel{1} - \cancel{1} + \frac{1}{N} = \left[1 - p\left(1 - \cancel{\frac{1}{N}}\right)\right]^{N-1} = [1-p]^{N-1}$$

for large N

$$\frac{1}{N} = [1-p]^{N-1} = e^{(N-1)\ln(1-p)} = e^{-p(N-1)} = e^{-\langle k\rangle}$$

$p \ll 1$ $\quad \ln(1-p) \simeq -p$

$$\frac{1}{N} = e^{-\langle k\rangle} \qquad \underbrace{\langle k\rangle = \ln N}$$

which is what we wanted to prove