WEEK 5 Lecture 2

4.5 POISSON NETWORKS

We have seen that in $G(N, p)$ ensemble the degree distribution is binomial

$$P_B(k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$$

$$k \sim \mathrm{Bin}(N-1, p)$$

Hence:

$$\langle k \rangle = (N-1)p$$

$$\langle k(k-1) \rangle = (N-1)(N-2)p^2$$

$$\langle k(k-1) \rangle = \langle k^2 \rangle - \langle k \rangle$$

$$\langle k^2 \rangle = \langle k(k-1) \rangle + \langle k \rangle \qquad \circledast$$

The variance is $\sigma^2 = \langle k^2 \rangle - \langle k \rangle^2 = \langle k(k-1) \rangle + \langle k \rangle - \langle k \rangle^2 =$

$$= (N-1)(N-2)p^2 + (N-1)p - (N-1)^2 p^2 =$$

$$= (N-1)p\left[(\cancel{N}-2)p + 1 - (\cancel{N}-1)p\right] =$$

$$= (N-1)p(1-p)$$

VARIANCE $\boxed{\sigma^2 = (N-1)p(1-p)}$

STANDARD DEVIATION $\boxed{\sigma = \sqrt{(N-1)p(1-p)}}$

PROPOSITION POISSON NETWORKS

The degree distribution of $G(N,p)$ ensemble with

$$p = \frac{c}{N-1}$$ where c is independent of N

can be approximated in the large network limit $(N \to \infty)$

by a Poisson distribution $P_p(k) = \dfrac{c^k e^{-c}}{k!}$

Proof is in the lecture notes NOT EXAMINABLE

- AVERAGE DEGREE of $G(N,p)$ when $p = \frac{c}{N-1}$ and $N \to \infty$

$$\langle k \rangle = p(N-1) = \frac{c}{\cancel{N-1}}\cancel{(N-1)} = c$$

$$p = \frac{c}{N-1}$$

AVERAGE DEGREE $\boxed{\langle k \rangle = c}$

- LINKING PROBABILITY

$$p = \frac{c}{N-1} \simeq \frac{c}{N}$$

$N \gg 1$

$$\langle k(k-1) \rangle = c^2$$

- STANDARD DEVIATION

Using ⊛ $\langle k^2 \rangle = \langle k(k-1) \rangle + \langle k \rangle = c^2 + c$

$$\sigma^2 = \langle k^2 \rangle - \langle k \rangle^2 = \cancel{c^2} + c - \cancel{c^2} = c$$

VARIANCE $\boxed{\sigma^2 = c}$

STANDARD DEVIATION $\boxed{\sigma = \sqrt{c}}$

- Mean and standard deviation of Binomial and Poisson degree distributions coincide

if $p = \dfrac{c}{N-1}$ and $N \to \infty$

	BINOMIAL	POISSON
MEAN	$\langle k \rangle = (N-1)\,p = (N-1)\dfrac{c}{N-1} = c$	$\langle k \rangle = c$
SECOND FACTORIAL MOMENT	$\langle k(k-1) \rangle = (N-1)(N-2)\,p^2 = \cancel{(N-1)}(N-2)\dfrac{c^2}{(N-1)^{\cancel{2}}} = \dfrac{N-2}{N-1}\,c^2 \xrightarrow[N\to\infty]{} c^2$	$\langle k(k-1) \rangle = c^2$

ok

ok

VARIANCE

$$\sigma^2 = (N-1)\,p\,(1-p) = \cancel{(N-1)}\,\frac{c}{\cancel{N-1}}\left(1-\frac{c}{N-1}\right) = c\left(1-\frac{c}{N-1}\right) \xrightarrow[N\to\infty]{} c$$

OK

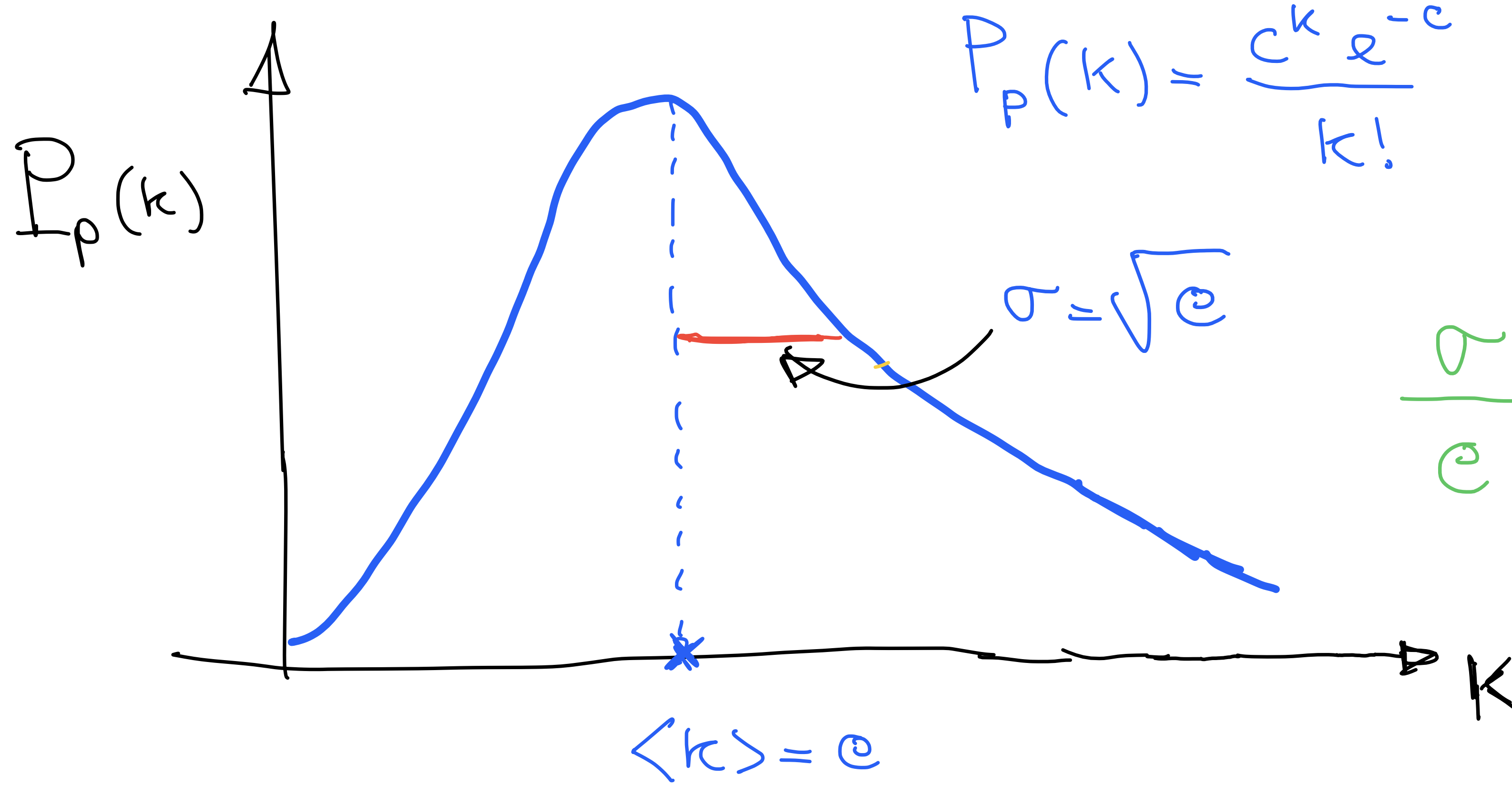
$$\sigma^2 = c$$

STANDARD DEVIATION

$$\sigma = \sqrt{\sigma^2} = \sqrt{c}$$

OK

$$\sigma = \sqrt{c}$$



$$P_p(k) = \frac{c^k e^{-c}}{k!}$$

$$\frac{\sigma}{c} = \frac{\sqrt{c}}{c} = \frac{1}{\sqrt{c}}$$

- The degree distribution of random graphs is HOMOGENEOUS while real-world networks have HETEROGENEOUS degree distribution

Social networks
airport networks
WWW

Let's calculate the probability of finding a node of degree $k = 1000$ in a Poisson network with average degree $\langle k \rangle = c = 100$

$$P_p(k) = \frac{c^k e^{-c}}{k!} = \frac{100^k e^{-100}}{k!}$$

To start let us notice that

$$\frac{|k - \langle k \rangle|}{\sigma} = \frac{1000 - 100}{10} = 90$$

$\sigma = \sqrt{\langle k \rangle}$

This indicates that a node with $K = 1000$ is 90 standard deviations away from the mean !!!!

$$P_p(1000) = \frac{100^{1000}\, e^{-100}}{1000!} = \dots = 10^{-609}$$

a very small number indeed !!

$$\log_{10} P = 10^3 \log 100 - 10^2 \log e - \log(1000!)$$

$$= 2 \cdot 10^3 - 0.434 \cdot 10^2 - 10^3 \log 10^3 + 10^3 \log e =$$

$$= \left[2 - 0.0434 - 3 + 0.434\right] 10^3 =$$

$$= -0.609 \cdot 10^3$$

$$P = 10^{-0.609 \cdot 10^3} = 10^{-609}$$

STIRLING'S approximation

$$\log_{10}(n!) = n \log n - n \log e$$