WEEK 4 tutorial

TAKE HOME MESSAGE: CENTRALITY

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

DEGREE

$$x_i = k_i^{IN} = \sum_{j=1}^{N} A_{ij}$$

EIGENVECTOR

$$x_i^{(n)} = \sum_{j=1}^{N} A_{ij} \, x_j^{(n-1)}$$

$$x_i^{(0)} = \frac{1}{N} \quad \forall i \qquad\qquad x_i = \lim_{n \to \infty} \frac{x_i^{(n)}}{\sum_j x_j^{(n)}}$$

For connected networks

$$\underline{x} \propto \underline{v}^1 \qquad\qquad A \, \underline{v}^1 = \lambda_1 \underline{v}^1$$

proportional

leading eigenvalue

P.F. 1

KATZ

$$x_i = \alpha \sum_{j=1}^{N} A_{ij} x_j + \beta$$

$\alpha \in \left(0, \frac{1}{\lambda_1}\right)$ $\beta > 0$

$$\underline{x} = \beta \left(\mathbb{I} - \alpha \mathbb{A}\right)^{-1} \underline{1} = \beta \sum_{n=0}^{\infty} (\alpha \mathbb{A})^n \underline{1}$$

PAGERANK

$$x_i = \alpha \sum_{j=1}^{N} A_{ij} \frac{x_j}{k_j^{out\,*}} + \beta$$

$\beta > 0$

$k_j^{out\,*} = \max\left(k_j^{out}, 1\right)$

$\alpha \in \left(0, \frac{1}{\mu_1}\right)$

μ_1: leading eigenvalue of $\mathbb{A}\mathbb{D}^{-1}$

$$\mathbb{D} = \begin{pmatrix} k_1^{out\,*} & & \\ & \ddots & \\ & & k_N^{out\,*} \end{pmatrix}$$

$$\underline{x} = \beta \left(\mathbb{I} - \alpha \mathbb{A}\mathbb{D}^{-1}\right)^{-1} \underline{1} = \beta \sum_{n=0}^{\infty} \left(\alpha \mathbb{A}\mathbb{D}^{-1}\right)^n \underline{1}$$

(2

BASED ON SHORTEST PATHS

CLOSENESS

$$x_i = \frac{1}{\frac{1}{N-1}\sum_{j=1}^{N} d_{ij}}$$

graph distances

EFFICIENCY

$$x_i = \frac{1}{N-1}\sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{1}{d_{ij}}$$

BETWEENNESS

$$x_i = \sum_{r=1}^{N}\sum_{s=1}^{N} \frac{n_{rs}^{i}}{g_{rs}}$$

# of s.p. from r to s passing by i

# of s.p. from r to s

From FA 3

Q 1

- **1. Centrality measures of a given directed network**
  Consider the adjacency matrix $\mathbf{A}$ of a directed network of size $N = 4$ given by

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In the following we will indicate with $\mathbf{1}$ the column vector $\mathbf{1}$ with elements $1_i = 1 \; \forall i = 1, 2 \ldots, N$ and we will indicate with $\mathbf{I}$ the identity matrix.

(a) Draw the network

(b) Calculate the eigenvector centrality using its definition. Is the result expected? (*Explain why*).

(c) Calculate the Katz centrality

$$\mathbf{x} = \beta(\mathbf{I} - \alpha\mathbf{A})^{-1}\mathbf{1} = \beta \sum_{n=0}^{\infty} \alpha^n \mathbf{A}^n \mathbf{1}. \tag{1}$$

(d) Calculate the PageRank centrality

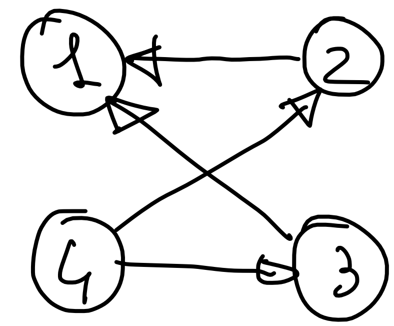
$$\mathbf{x} = \beta(\mathbf{I} - \alpha\mathbf{A}\mathbf{D}^{-1})^{-1}\mathbf{1} = \beta \sum_{n=0}^{\infty} \alpha^n [\mathbf{A}\mathbf{D}^{-1}]^n \mathbf{1} \tag{2}$$

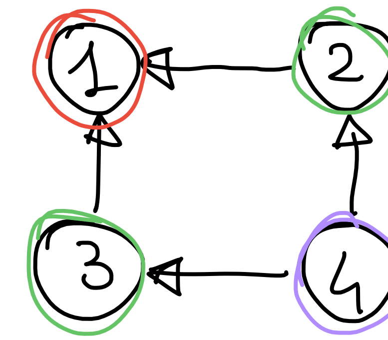
where $\mathbf{D}$ is a diagonal matrix with non-zero elements $D_{ii} = \kappa_i = \max(k_i^{out}, 1)$ and k_i^{out} is the out-degree of node i.

(a)

$$\mathbb{A} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$N=4 \quad L=4$ directed





b Eigenvector

democratic guess

$$x_i^{(0)} = \frac{1}{4} \qquad \forall i = 1,2,3,4$$

$$\underline{x}^{(1)} = \mathbb{A}\,\underline{x}^{(0)} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/4 \\ 1/4 \\ 0 \end{pmatrix}$$

$$\underline{x}^{(2)} = \mathbb{A}\,\underline{x}^{(1)} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/4 \\ 1/4 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\underline{x}^{(3)} = \mathbb{A}\,\underline{x}^{(2)} = \dots = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \underline{x}^{(m)} = 0 \quad \forall m \geq 3$$

It follows that $\underline{x} = 0$

as expected as the graph is not strongly connected

(c) KATZ

$$\underline{x} = \beta \sum_{n=0}^{\infty} \alpha^n \mathbb{A}^n \underline{1}$$

$$\mathbb{A}^0 = \mathbb{I} \qquad \mathbb{A}^1 = \mathbb{A} \qquad \mathbb{A}^2 = \mathbb{A}\cdot\mathbb{A} = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbb{A}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underline{x} = \beta \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \alpha \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \alpha^2 \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} =$$

$$= \beta \begin{pmatrix} 1 + 2\alpha + 2\alpha^2 \\ 1 + \alpha \\ 1 + \alpha \\ 1 \end{pmatrix} \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix}$$

$$x_4 < x_3 = x_2 < x_1$$

$$\beta = \frac{1}{\sum_{i=1}^{4} x_i}$$

(d) PAGERANK

$$\underline{x} = \beta \sum_{n=0}^{\infty} \alpha^n \left(\mathbb{A} \mathbb{D}^{-1} \right)^n \underline{1}$$

$$K_i^{OUT} = \{0, 1, 1, 2\} \qquad K_i^{OUT^*} = \{1, 1, 1, 2\}$$

$$\mathbb{D} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \qquad \mathbb{D}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\mathbb{A}\mathbb{D}^{-1} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(\mathbb{A}\mathbb{D}^{-1})^2 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(\mathbb{A}\mathbb{D}^{-1})^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(\mathbb{A}\mathbb{D}^{-1})^n = 0 \qquad \forall n \geq 3$$

$$\underline{x} = \beta \left[\mathbb{I} + \alpha \, \mathbb{A}\mathbb{D}^{-1} + \alpha^2 (\mathbb{A}\mathbb{D}^{-1})^2 \right] \underline{1}$$

$$= \beta \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \alpha \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \alpha^2 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} =$$

$$= \beta \begin{array}{cc} \begin{pmatrix} 1 + 2\alpha + \alpha^2 \\ 1 + \frac{1}{2}\alpha \\ 1 + \frac{1}{2}\alpha \\ 1 \end{pmatrix} & \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \end{array}$$

$$x_4 < x_3 = x_2 < x_1$$

Important Info $\rightarrow$ QUIZ 2 opens today at 6 PM

8