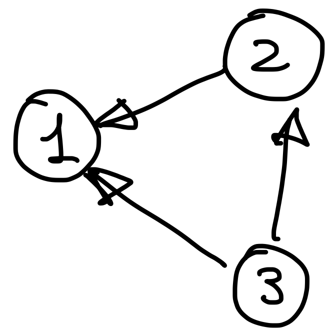
Week 4 Lecture 2

Ex

G_2



$$\mathbb{A} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Compute the PageRank centrality $\underline{x} = \beta \sum_{n=0}^{\infty} \left(\alpha \mathbb{A} \mathbb{D}^{-1} \right)^n \underline{1}$

Out-degree sequence $\{0, 1, 2\}$ $\{1, 1, 2\}$

$$k_i^{OUT\,*} = \max\{k_i^{OUT}, 1\}$$

D_{ii}

$$\mathbb{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \mathbb{D}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$$

$$\mathbb{A}\mathbb{D}^{-1} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$(\mathbb{A}\mathbb{D}^{-1})^0 = \mathbb{I}$$

$$(\mathbb{A}\,\mathbb{D}^{-1})^1 = \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$(\mathbb{A}\,\mathbb{D}^{-1})^2 = \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(\mathbb{A}\,\mathbb{D}^{-1})^3 = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (\mathbb{A}\,\mathbb{D}^{-1})^n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \forall n \geq 3$$

$$\underline{x} = \beta \sum_{n=0}^{\infty} (\alpha\, \mathbb{A}\,\mathbb{D}^{-1})^n\, \underline{1}$$

$$= \beta \left[\overset{\mathbb{I}}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} + \alpha \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} + \alpha^2 \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} =$$

$$= \beta \left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} + \alpha^2 \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \right] = \beta \begin{pmatrix} 1 + \frac{3}{2}\alpha + \frac{1}{2}\alpha^2 \\ 1 + \frac{1}{2}\alpha \\ 1 \end{pmatrix} \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}$$

$$x_1 > x_2 > x_3$$

fix β: $\sum_{i=1}^{3} x_i = 1$

$$\beta = \frac{1}{x_1 + x_2 + x_3} = \frac{1}{3 + 2\alpha + \frac{1}{2}\alpha^2}$$

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3.7 CLOSENESS and EFFICIENCY CENTRALITY

IDEA: A node is central if it is at short distance from other nodes

DEF CLOSENESS

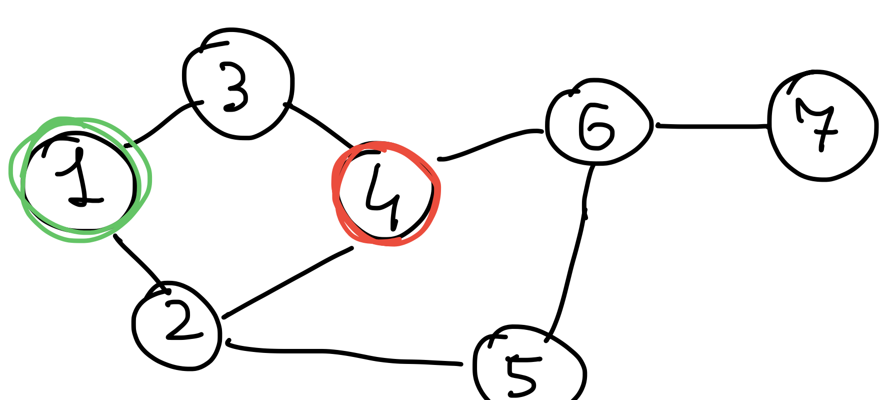
Let G be a connected network

The closeness centrality Cl_i of node i is:

$$Cl_i = \frac{1}{\frac{1}{N-1}\sum_{j=1}^{N} d_{ij}} = \frac{1}{\ell_i}$$

average (shortest-path) distance from i to other nodes

EX



$N = 7$

Compute Cl_1 $\quad d_{12} = d_{13} = 1 \quad d_{14} = d_{15} = 2 \quad d_{16} = 3 \quad d_{17} = 4$

$$Cl_1 = \frac{\overset{6}{N-1}}{\underbrace{1+1+2+2+3+4}_{N-1=6 \text{ terms}}} = \frac{6}{13}$$

Compare to Cl_4

$$d = \begin{array}{c|ccccccc|c} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\ \hline & 0 & 1 & 1 & 2 & 2 & 3 & 4 & 1 \\ & - & - & - & - & - & - & - & 2 \\ & - & - & - & - & - & - & - & 3 \\ & 2 & 1 & 1 & 0 & 2 & 1 & 2 & 4 \\ & & & & & & & & 5 \\ & & & & & & & & 6 \\ & & & & & & & & 7 \end{array}$$

$$Cl_4 = \frac{6}{2+1+1+2+1+2} = \frac{6}{9} = \frac{2}{3} > Cl_1$$

DISADVANTAGES

$Cl_i = 0 \quad \forall i$

- If the network is not connected we cannot use it
- Small range of variability due to the SWDP

DEF EFFICIENCY

CATORA, MARCHIORI 2001

The efficiency centrality E_i of node i is:

$$E_i = \frac{1}{N-1} \sum_{J=1}^{N} \frac{1}{d_{iJ}}$$

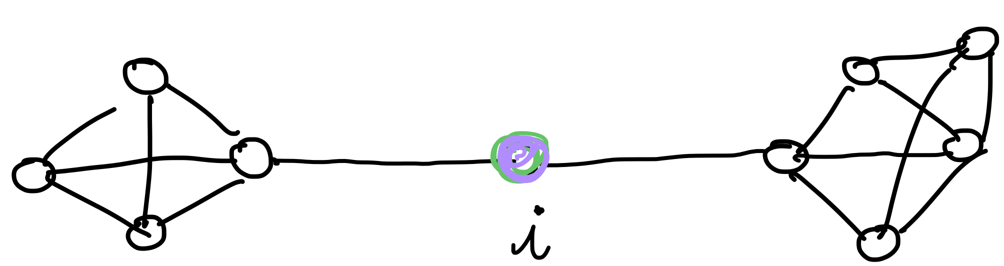
this is the efficiency in the communication between i and J

also works for unconnected networks

3.8 BETWEENNESS CENTRALITY

IDEA: A node is central if it connects many pairs of nodes

if it is traversed by many shortest paths



i has low degree but high betweenness

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FREEMAN

$\boxed{\text{DEF}}$ BETWEENNESS

The betweenness centrality b_i of node i is:

$$b_i = \sum_{r=1}^{N} \sum_{s=1}^{N} \frac{n_{rs}^{i}}{g_{rs}}$$

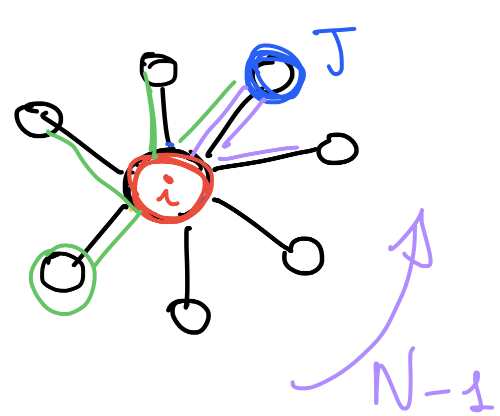
# of shortest paths from r to s PASSING by i

# of shortest paths going from r to s

SIMPLIFIED VERSION

EX

Star of N nodes



Notice: there is only 1 shortest path for each pair of nodes

Simplify the calculation

$$g_{rs} = 1 \quad \forall r, s$$

- A leaf node J

$$b_J = N + N - 1 = 2N - 1$$

# of paths that start from J and end in any other node including J

# of paths that start from any node, different from J, and end in J

- The centre of the star i

$$b_i = N + N - 1 + (N-1)(N-2) =$$

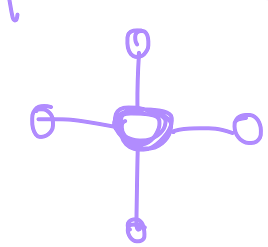
# of paths starting from i

# of paths arriving at i

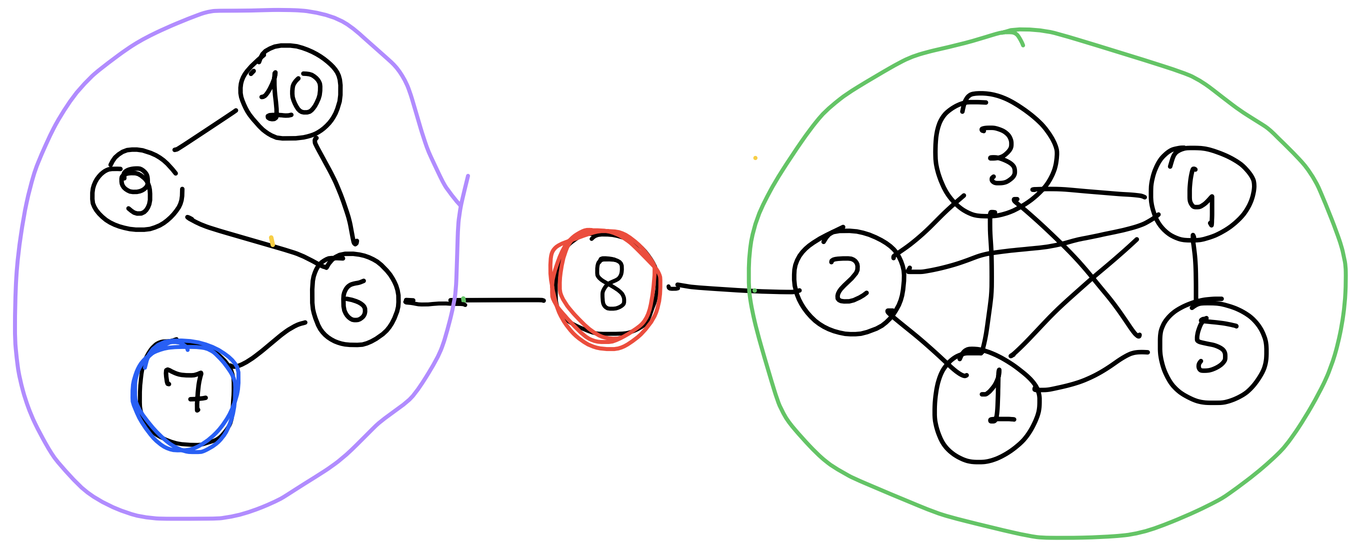
# of paths from a leaf to another leaf

$$= N + N - 1 + N^2 - 3N + 2 = N^2 - N + 1$$

Try for $N = 5$



Ex



$N = 10$

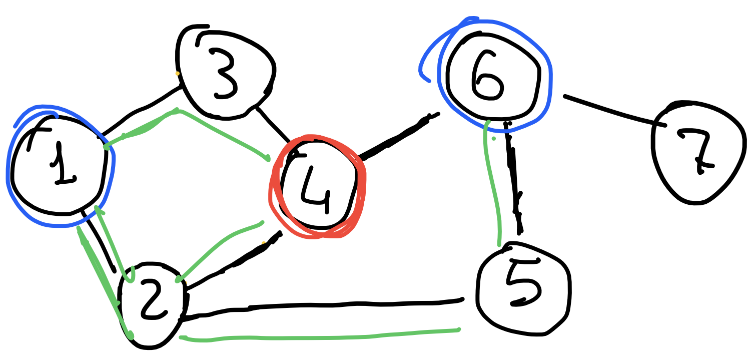
$$b_8 = 10^2 - 4^2 - 5^2 = 59$$

$$b_7 = 2N - 1 = 19$$

7 is a leaf

Ex

$N = 7$ b_4



matrix for node 4

	1	2	3	4	5	6	7
1	0	0	0	1	0	2/3	2/3
2			1/2	1		1/2	1/2
3		1/2		1	2/3		
4	1	1	1	1	1	1	1
5			2/3	1			
6	2/3	1/2	1	1			
7	2/3	1/2	1	1			

From 1 to 6 I have 3 shortest paths and only 2 of them are passing by node 4

To calculate b_4 you need to evaluate all the entries of the matrix above and then sum them up