WEEK 4 Lecture 1

CHAPTER 3
CENTRALITY MEASURES

3.1 INTRODUCTION

Important nodes have strategic locations in the network

individuals in a social network

centrality measures

x_i

3.2 DEGREE CENTRALITY

IDEA: nodes with high DEGREE, the "hubs" of the network are more active and hence more central

Undirected networks

$x_i = k_i$

Directed networks

$$x_i = \begin{cases} k_i^{IN} \\ k_i^{OUT} \end{cases}$$

2 possibilities

the # of nodes pointing to node i

3.3 EIGENVECTOR (BONACICH) CENTRALITY

IDEA: A node is central if central nodes point to it

DEF EIGENVECTOR CENTRALITY

The eigenvector centrality $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$ can be obtained by iteration:

- initial guess $x_i^{(0)} \quad i = 1, 2, \dots, N$
- recursive process

$$x_i^{(n)} = \sum_{j=1}^{N} A_{ij}\, x_j^{(n-1)}$$

ⓐ If $\exists \lim_{n\to\infty} \sum_{j=1}^{N} x_j^{(n)} > 0$ then the eigenvector centrality of node i is $x_i = \lim_{n\to\infty} \frac{x_i^{(n)}}{\sum_j x_j^{(n)}}$

ⓑ If $\underbrace{\exists \lim_{n\to\infty} \sum_{j=1}^{N} x_j^{(n)} = 0}$ then $x_i = 0 \quad \forall i \in \{1, 2, \dots, N\}$

The typical initial guess is the so-called "democratic guess"

$$x_i^{(0)} = \frac{1}{N} \quad \forall i \in \{1, 2, \dots, N\}$$

However if $\lim_{n\to\infty} x_i^{(n)}$ does not exist for some i we need to take another initial guess

We are "covered" by the PERRON-FROBENIUS Theorem

see Gnestre's lecture notes pag 56

PROPOSITION

In a connected undirected network and in directed network with a (non-trivial) strongly-connected component the eigenvector centrality $\underline{x}$ is proportional to $\underline{v}^1$ (i.e. $\underline{x} = c\, \underline{v}^1$)

where $\underline{v}^1$ is the eigenvector associated to the leading eigenvalue of $\mathbb{A}$:

$$\mathbb{A}\, \underline{v}^1 = \lambda_1 \underline{v}^1$$

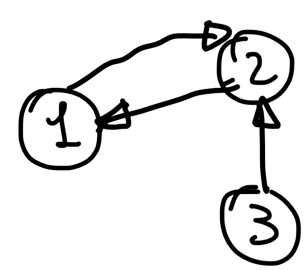
c is fixed by the condition $\sum_i x_i = 1$

by Perron-Frobenius λ_1 is real and positive with $\lambda_1 \geq |\lambda_i| \ \forall i$ and has multiplicity 1

and $x_i \geq 0 \quad \forall i$

Ex

$$\mathbb{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$



Network G_1

has a SCC $\{1, 2\}$ non-trivial

Let us calculate $\underline{v}^1$ associated to λ_1

$$\mathbb{A}\, \underline{v} = \lambda\, \underline{v}$$

$$0=\det(A-\lambda I)=\det\begin{pmatrix}-\lambda & 1 & 0\\ 1 & -\lambda & 1\\ 0 & 0 & -\lambda\end{pmatrix}=-\lambda^3+\lambda=-\lambda(\lambda^2-1)\to\begin{cases}\lambda_1=1\\ \lambda_2=0\\ \lambda_3=-1\end{cases}$$

$$\lambda_1>\lambda_2>\lambda_3 \qquad \lambda_1=1$$

$$A\underline{v}=\underline{v} \qquad (A-I)\underline{v}=0 \qquad \begin{pmatrix}-1 & 1 & 0\\ 1 & -1 & 1\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix}=0$$

$$\begin{cases}-v_1+v_2=0\\ v_1-v_2+v_3=0\\ -v_3=0\end{cases} \qquad \begin{cases}v_1=v_2\\ \\ v_3=0\end{cases} \qquad \underline{v}^1=\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$\underline{x}=c\begin{pmatrix}1\\1\\0\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}1/2\\1/2\\0\end{pmatrix}$$

centrality

$c: \sum_i x_i=1$

REMARK

$$A\underline{x}=\lambda_1\underline{x} \qquad \alpha=\frac{1}{\lambda_1}$$

$$\underline{x}=\frac{1}{\lambda_1}A\underline{x}$$

$$x_i=\frac{1}{\lambda_1}\sum_j A_{ij}x_j$$

α

Hence a node is central if central nodes point to it

Ex Alternatively we can use the iterative process to calculate $\underline{x}$

Initial guess

A of G_1

We get oscillations

$$\underline{x}^{(0)} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} \qquad \underline{x}^{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \\ 0 \end{pmatrix} \qquad \underline{x}^{(2)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/3 \\ 2/3 \\ 0 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \\ 0 \end{pmatrix}$$

$$\underline{x}^{3} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2/3 \\ 1/3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \\ 0 \end{pmatrix} \qquad \underline{x}^{4} = \dots = \begin{pmatrix} 2/3 \\ 1/3 \\ 0 \end{pmatrix}$$

Case (a)

Notice that $\lim_{n\to\infty} \sum_i x_i^{(n)} = 1$ (a) is verified

However $\nexists \lim_{n\to\infty} x_1^{(n)}$ so we need to try a different initial guess

$$\underline{x}^{(0)} = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} \qquad \underline{x}^{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} = \underline{x}^{(2)} = \dots = \underline{x}^{(n)}$$

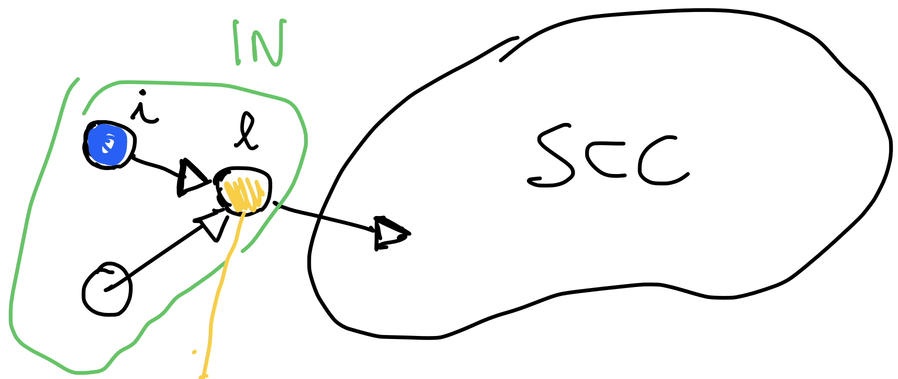
centrality

$$\underline{x} = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}$$

PROPOSITION

In a directed network all the nodes in the IN-COMPONENT of a SCC have centrality $x_i = 0$

Proof



Leaf node i has $x_i = 0$ because $x_i^{(n)} = \sum_j A_{ij} x_j^{(n-1)}$

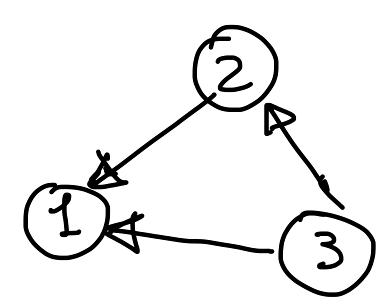
$$x_i^{(1)} = \sum_j \underbrace{A_{ij}}_{=0 \ \forall j} x_j^{(0)} = 0$$

because i is a leaf

The same thing will happen to node ℓ at iteration 2 (i.e. $x_\ell^{(2)} = 0$)

Ex

$$\mathbb{A} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$



G_2

No (non-trivial) SCC

Initial guess

$$\underline{x}^{(0)} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} \qquad \underline{x}^{(1)} = \mathbb{A}\,\underline{x}^{(0)} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \\ 0 \end{pmatrix}$$

$$\underline{x}^{(2)} = \mathbb{A}\, \underline{x}^{(1)} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2/3 \\ 1/3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 0 \\ 0 \end{pmatrix}$$

$$\underline{x}^{(3)} = \mathbb{A}\, \underline{x}^{(2)} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \underline{x}^{(n)} = \underline{0} \quad \forall n \geq 3$$

CASE b

$\lim_{n \to \infty} \sum_J x_J^{(n)} = 0 \Rightarrow$ by definition we have $\underline{x} = \underline{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

It is not a "nice" result that $x_i = 0 \;\; \forall i$ in G_2

because $\quad k_1^{IN} = 2 \qquad k_2^{IN} = 1 \qquad k_3^{IN} = 0$

3.4 KATZ CENTRALITY

Katz solved this problem

IDEA : Add a fixed (non-zero) centrality to each node

DEF KATZ CENTRALITY

The Katz centrality $\underline{x}$ satisfies

$$x_i = \alpha \sum_{j=1}^{N} A_{ij} x_j + \beta$$

where $\beta > 0$ and $\alpha \in \left(0, \frac{1}{\lambda_1}\right)$ leading eigenvalue of $\mathbb{A}$

In matrix formalism

$$\underline{x} = \alpha \mathbb{A} \underline{x} + \beta \underline{1}$$

N-dimensional vector of elements

$$1_i = 1 \quad i = 1, \dots, N$$

$$\underline{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$\mathbb{I} \underline{x} - \alpha \mathbb{A} \underline{x} = \beta \underline{1}$$

$$\left(\mathbb{I} - \alpha \mathbb{A}\right) \underline{x} = \beta \underline{1}$$

If $\det\left(\mathbb{I} - \alpha \mathbb{A}\right) \neq 0$ we can invert the matrix

if $\frac{1}{\alpha} \neq \lambda$ $\qquad \alpha \neq \frac{1}{\lambda}$

eigenvalues of A

This is satisfied if $0 < \alpha < \frac{1}{\lambda_1}$

largest eigenvalue of $\mathbb{A}$

$$\boxed{\underline{x} = \beta \left(\mathbb{I} - \alpha \mathbb{A} \right)^{-1} \underline{1} = \beta \sum_{n=0}^{\infty} (\alpha \mathbb{A})^n \underline{1}}$$

CONVERGENT for $|\alpha \lambda_1| < 1$

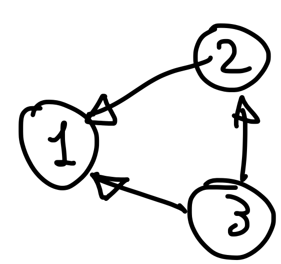
MATRIX SERIES $\mathbb{I} + \alpha \mathbb{A} + \alpha^2 \mathbb{A}^2 + \dots = \sum_{n=0}^{\infty} (\alpha \mathbb{A})^n = \left(\mathbb{I} - \alpha \mathbb{A} \right)^{-1}$

extension of geometric series $\sum_{n=0}^{\infty} (\alpha z)^n = \frac{1}{1 - \alpha z}$

CONVERGENT for $|\alpha z| < 1$

Ex

G_2



$$\mathbb{A} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Compute Katz centrality

$\mathbb{A}^0 = \mathbb{I}$ $\quad \mathbb{A}^1 = \mathbb{A}$ $\quad \mathbb{A}^2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\mathbb{A}^3 = \mathbb{A}^2 \mathbb{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \mathbb{A}^n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ \forall n \geq 3$

$$\underline{x} = \beta \sum_{n=0}^{\infty} (\alpha \mathbb{A})^n \underline{1} = \beta \left(\mathbb{I} + \alpha \mathbb{A} + \alpha^2 \mathbb{A}^2 \right) \underline{1}$$

$$= \beta \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \alpha \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \alpha^2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} =$$

$$= \beta \left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \alpha^2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] = \beta \begin{pmatrix} 1 + 2\alpha + \alpha^2 \\ 1 + \alpha \\ 1 \end{pmatrix} \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}$$

NORMALIZATION FACTOR

$$\beta = (x_1 + x_2 + x_3)^{-1} =$$

$$= (3 + 3\alpha + \alpha^2)^{-1}$$

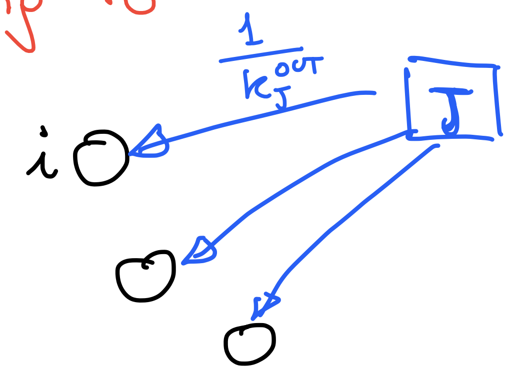
$$\underline{x_3 < x_2 < x_1}$$

Alternatively you can calculate Katz centrality inverting matrix $\mathbb{I} - \alpha \mathbb{A}$

3.5 PAGERANK CENTRALITY

The famous algorithm used by GOOGLE to rank webpages

IDEA
1. a node is central if pointed by other central nodes ← eigenvector
2. there is a fixed (non-zero) centrality for each node ← Katz
3. A node divides its centrality (equally) to all nodes it is pointing to



DEF PAGERANK CENTRALITY

The PageRank centrality $\underline{x}$ satisfies

$$x_i = \alpha \sum_{J=1}^{N} A_{iJ} \frac{x_J}{k_J^{OUT*}} + \beta$$

where $k_J^{OUT*} = \max\left(k_J^{OUT}, 1\right)$, $\beta > 0$ and $\alpha \in \left(0, \frac{1}{\mu_1}\right)$

μ_1 is the leading eigenvalue of matrix $\mathbb{A}\mathbb{D}^{-1}$

$\mathbb{D}$ is the diagonal matrix with elements $D_{JJ} = k_J^{out\,*}$

In matrix formalism

$$\underline{x} = \alpha \mathbb{A} \mathbb{D}^{-1} \underline{x} + \beta \underline{1}$$

$$\left(\mathbb{I} - \alpha \mathbb{A} \mathbb{D}^{-1}\right) \underline{x} = \beta \underline{1}$$

If $\det\left(\mathbb{I} - \alpha \mathbb{A} \mathbb{D}^{-1}\right) \neq 0$ you can invert matrix $\mathbb{I} - \alpha \mathbb{A} \mathbb{D}^{-1}$

It is sufficient to assume $0 < \alpha < \frac{1}{\mu_1}$

largest eigenvalue of $\mathbb{A}\mathbb{D}^{-1}$

$$\boxed{\underline{x} = \beta \left(\mathbb{I} - \alpha \mathbb{A} \mathbb{D}^{-1}\right)^{-1} \underline{1} = \beta \sum_{n=0}^{\infty} \left(\alpha \mathbb{A} \mathbb{D}^{-1}\right)^{n} \underline{1}}$$