Queen Mary
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Complex Networks (MTH6142)
Solutions of Formative Assignment 2

- **1*** **Average degree of a growing network**

  Assume that you are observing a growing undirected network.
  The network evolves in time by the following rules:
  At time $t = 1$ there is a single isolated node.
  At each time $t > 1$ a new node is added to the network and is connected to the existing network by a new link.
  Consider the network at time $t = T$.

  (a) What is the total number of nodes N?
  (b) What is the total number of links L?
  (c) What is the average degree $\langle k \rangle$?
  (d) What is the average degree in the limit $T \to \infty$?

- *Notes on solution*

  (a) The total number of nodes is $N = T$.
  In fact we start from time $t = 1$ with a single node and at each time $t > 1$ we add a new node.
  (b) The total number of links is $L = T - 1$.
  In fact at time $t = 1$ there are no links and at each time $t > 1$ we add a new link.
  (c) The average degree $\langle k \rangle$ is given by

  $$\langle k \rangle = \frac{2L}{N} = 2\frac{T-1}{T} = 2\left(1 - \frac{1}{T}\right). \tag{1}$$

  (d) In the limit $T \to \infty$ we have

  $$\lim_{T\to\infty} \langle k \rangle = \lim_{T\to\infty} 2\left(1 - \frac{1}{T}\right) = 2. \tag{2}$$

- **2. Matrix Formalism.**
  Consider a simple network of size N. Let $\mathbf{A}$ be the $N \times N$ adjacency matrix and let $\mathbf{1}$ be the N dimensional column vector whose elements are given by $1_i = 1\ \forall i = 1, 2, \ldots N$, and $\mathbf{k}$ be the N dimensional column vector whose elements are given by the degrees $k_i\ \forall i = 1, 2, \ldots, N$, i.e.

  $$\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \qquad \mathbf{k} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_N \end{pmatrix} \tag{3}$$

Using the matrix formalism (row by column product) show that

(a) the vector $\mathbf{k}$ whose elements are the degrees k_i of the nodes $i = 1, 2, \ldots, N$ can be written as

$$\mathbf{k} = \mathbf{A}\mathbf{1}. \tag{4}$$

(b) the number L of links in the network can be written as

$$L = \frac{1}{2}\mathbf{1}^T\mathbf{A}\mathbf{1} \tag{5}$$

(c) the matrix $\mathbf{N}$ whose element N_{ij} is equal to the number of common neighbours of nodes i and j can be written as

$$\mathbf{N} = \mathbf{A}^{\mathbf{2}} \tag{6}$$

- *Notes on solution*

(a) The degree of a node in a simple network is given by

$$k_i = \sum_{j=1}^{N} A_{ij} = \sum_{j=1}^{N} A_{ij} 1_j. \tag{7}$$

It follow that the vector $\mathbf{k}$ can be expressed as

$$\mathbf{k} = \mathbf{A}\mathbf{1}. \tag{8}$$

(b) The total number of links the network is given by

$$L = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} A_{ij} = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} 1_i A_{ij} 1_j. \tag{9}$$

It follows that L can then be written as

$$L = \frac{1}{2}\mathbf{1}^T\mathbf{A}\mathbf{1}. \tag{10}$$

(c) A node r is a common neighbour of both node i and node j if and only if $A_{ir}A_{rj} = 1$. Therefore the total number of common neighbor of node i and node j is given by $N_{ij} = \sum_{r=1}^{N} A_{ir}A_{rj}$. Therefore we have

$$\mathbf{N} = \mathbf{A}^2. \tag{11}$$

- $3^{\star}$. **Diameter of simple networks.**
One can calculate the diameter of certain types of network exactly. Assume that each of these network has network size N.

(a) What is the diameter of a fully connected network?

(b) What is the diameter of a star network?

(c) What is the diameter of a linear chain of N nodes? (see figure 1)

(d) What is the diameter D of a square portion of square lattice, with l nodes along each side (see figure 1) ?

(e) Consider the expression found in question (3d) and find the leading term of D in terms of the total number of nodes N in the network, in the limit $N \gg 1$ proving that for $N \gg 1$

$$D \simeq 2\sqrt{N}. \tag{12}$$

(f) Which of the above networks have the small-world distance property?

- *Notes on the solution*

(a) The fully connected network is a network in which every node is liked to every other node. Therefore any two nodes are at distance $d = 1$. If follows that the diameter of the network is $D = 1$.

(b) A star network is a network in which a central node is linked to $N-1$ leaves nodes of degree 1. Any two leaves of the network are at distance $d = 2$ while the central node is at distance $d = 1$ from any leaf node. Therefore the diameter of the star network is $D = 2$.

(c) In a chain of N nodes the maximum distance is the distance between the two nodes at the edges of the chain. Therefore $D = N - 1$.

(d) In a square portion of a square lattice, the most distant pairs of nodes are formed by the nodes at the two opposite corners of the square lattice. These nodes are at distance $d = 2(l - 1)$, therefore the diameter of the network is

$$D = 2(l - 1) \tag{13}$$

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(e) Since the number of nodes $N = l^2$ we have $D = 2(\sqrt{N} - 1)$. In the limit $N \gg 1$ we have

$$D \simeq 2\sqrt{N}. \tag{14}$$

(f) A network displays the small-world distance property if its diameter D satisfies

$$D = \mathcal{O}(\ln N) \tag{15}$$

or

$$D = o(\ln N) \tag{16}$$

or equivalently if

$$\lim_{N\to\infty} \frac{D}{\ln N} < \infty \tag{17}$$

(i) The fully connected network has diameter

$$D = 1 \ \forall N. \tag{18}$$

Therefore

$$\lim_{N\to\infty} \frac{D}{\ln N} = 0, \tag{19}$$

and the network has the small world distance property.

(ii) The star network has diameter

$$D = 2 \ \forall N. \tag{20}$$

Therefore

$$\lim_{N\to\infty} \frac{D}{\ln N} = 0, \tag{21}$$

and the network has the small world distance property.

(iii) The diameter of a linear chain network is $D = N - 1 \gg$ for $N \gg 1$, therefore

$$\lim_{N\to\infty} \frac{D}{\ln N} = \infty, \tag{22}$$

i.e. the chain *does not* have the small world distance property.

(iv) The square portion of the square lattice has a diameter going like

$$D \simeq 2\sqrt{N} \tag{23}$$

for $N \gg 1$, therefore

$$\lim_{N\to\infty} \frac{D}{\ln N} = \infty, \tag{24}$$

i.e. the network *does not* have the small world distance property.

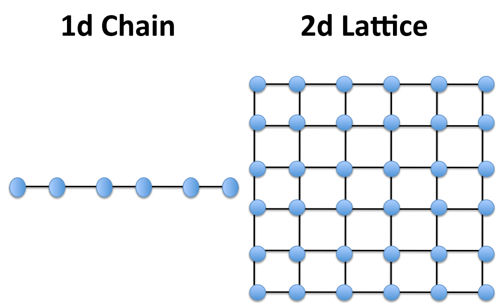

Figure 1: A linear chain network and a square portion of a square lattice with $l = 6$ nodes along each side.