# WEEK 3 Tutorial

## TAKE HOME MESSAGES FROM W 3

Def: Walks, Trails, Paths

Circuits, Cyclic paths

shortest paths → graph distance $\begin{cases} \text{average distance } \ell \\ \text{diameter } D \end{cases}$

shortest paths → graph components

SWDP: $\lim_{N \to \infty} \frac{D}{\ln N} = c < \infty$

also $c = 0$

1

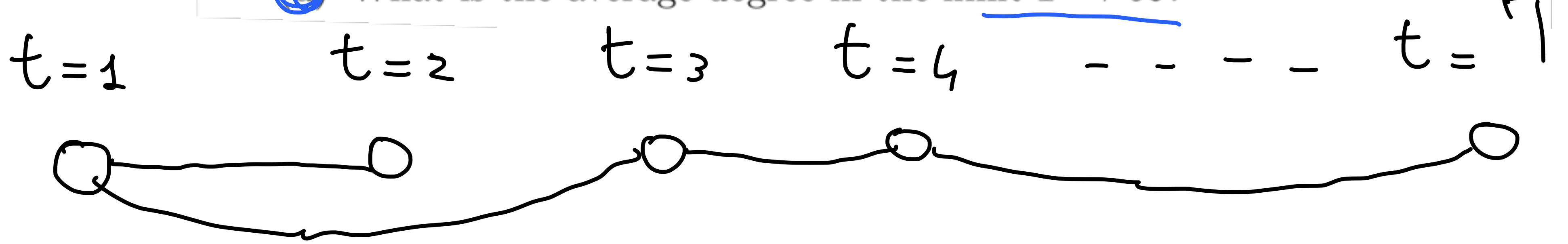
From FA2

Q1

- **1* Average degree of a growing network**

  Assume that you are observing a growing undirected network.
  The network evolves in time by this simple rules:
  *At time $t = 1$ there is a single isolated node.*
  *At each time $t > 1$ a new node is added to the network and is connected to the existing network by a new link.*
  Consider the network at time $t = T$.

  (a) What is the total number of nodes $N$?
  (b) What is the total number of links $L$?
  (c) What is the average degree $\langle k \rangle$?
  (d) What is the average degree in the limit $T \to \infty$?




(a) $N = T$

(b) $L = T - 1$

ⓒ $\langle k \rangle = \dfrac{2L}{N} \approx 2\,\dfrac{T-1}{T} = 2\left(1 - \dfrac{1}{T}\right)$

from W2

ⓓ $\lim\limits_{T \to \infty} \langle k \rangle = \lim\limits_{T \to \infty} 2\left(1 - \dfrac{1}{T}\right) = 2$

Q2

- **2. Matrix Formalism.**
  Consider a simple network of size $N$. Let $\mathbf{A}$ be the $N \times N$ adjacency matrix and let $\mathbf{1}$ be the $N$ dimensional column vector whose elements are given by $1_i = 1\ \forall i = 1, 2, \ldots N$, and $\mathbf{k}$ be the $N$ dimensional column vector whose elements are given by the degrees $k_i\ \forall i = 1, 2, \ldots, N$, i.e.

$$\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_N \end{pmatrix} \tag{1}$$

  Using the matrix formalism (row by column product) show that

  (a) the vector $\mathbf{k}$ whose elements are the degrees $k_i$ of the nodes $i = 1, 2, \ldots, N$ can be written as

$$\mathbf{k} = \mathbf{A}\mathbf{1}. \tag{2}$$

  (b) the number $L$ of links in the network can be written as

$$L = \frac{1}{2}\mathbf{1}^T \mathbf{A}\mathbf{1} \tag{3}$$

  (c) the matrix $\mathbf{N}$ whose element $N_{ij}$ is equal to the number of common neighbours of nodes $i$ and $j$ can be written as

$$\mathbf{N} = \mathbf{A}^2 \tag{4}$$

a Degree of node $i$ $\quad k_i = \sum_j A_{ij} = \sum_j A_{ij} \cdot 1_j \quad \forall i$

from W2

$$\begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_N \end{pmatrix} = \begin{pmatrix} A_{11} & \cdots \\ \vdots & \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$\underline{k} = A\,\underline{1}$$

4

(b) $L = \frac{1}{2}\sum_i \sum_j A_{ij} = \frac{1}{2}\sum_i \sum_j 1_i A_{ij} 1_j$

from W2

$k_i$

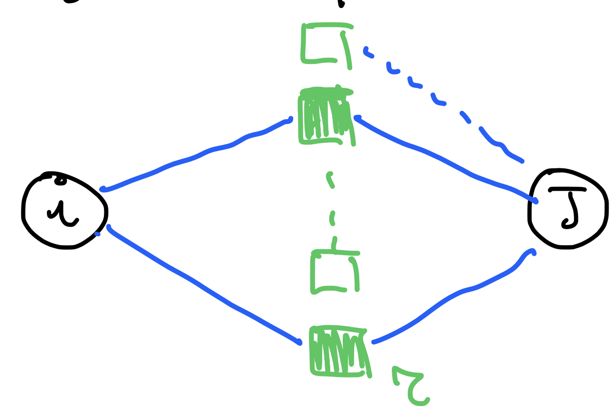
$$\frac{1}{2}\underline{1}^T A \underline{1}$$

In alternative use

$L = \frac{1}{2}\sum_i k_i$

....

(c) $N_{ij} \equiv$ # of common neighbours of $i$ and $j$ =




= # of walks of length 2 between $i$ and $j$ =

$$= \sum_r A_{ir} A_{rj}$$

$$N = A^2$$

$\boxed{\text{Q3}}$

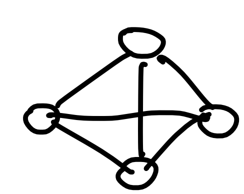
- **3*. Diameter of simple networks.**
  One can calculate the diameter of certain types of network exactly. Assume that each of these network has network size $N$.

  (a) What is the diameter of a fully connected network?
  (b) What is the diameter of a star network?
  (c) What is the diameter of a linear chain of $N$ nodes? (see figure 1)
  (d) What is the diameter $D$ of a square portion of square lattice, with $l$ nodes along each side (see figure 1 ) ?
  (e) Consider the expression ~~find~~ found in question (3d) and find the leading term of $D$ in terms of the total number of nodes $N$ in the network, in the limit $N \gg 1$ proving that for $N \gg 1$
  $$D \simeq 2\sqrt{N}. \tag{5}$$
  SWDP
  (f) Which of the above networks have the small-world distance property?

(a) Fully connected network

"Complete"

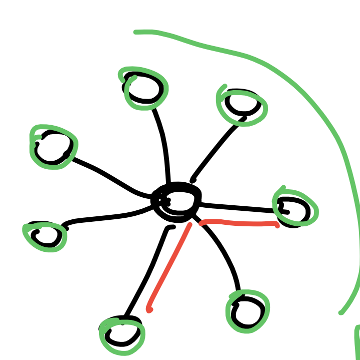
$N = 4$



$D = 1$ $\forall N$

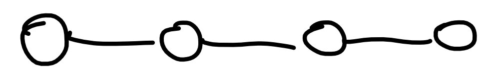
(b) Star network

with $N$ nodes



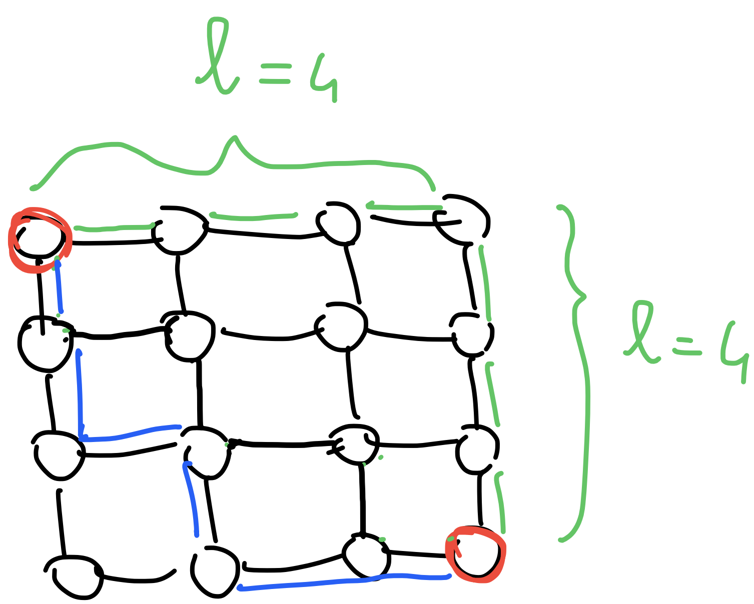
$N-1$ nodes

$D = 2$ $\forall N$

(c) Linear chain $N=4$  $D=3$

$N$  $D = N-1$

(d) 2D finite lattice




$N = \ell \cdot \ell \qquad \ell = \sqrt{N}$

$$D = 2(\ell - 1) = 2(\sqrt{N} - 1)$$

$$D = 2(\sqrt{N} - 1)$$

(e) $D = 2(\sqrt{N} - 1) \underset{N \gg 1}{\simeq} 2\sqrt{N} = 2N^{\frac{1}{2}}$

(f)

Star : $D = 2 \quad \forall N$

$$\lim_{N \to \infty} \frac{D}{\ln N} = \lim_{N \to \infty} \frac{2}{\ln N} = 0$$

YES SWDP

Square lattice : $D = 2(\sqrt{N} - 1)$

$$\lim_{N \to \infty} \frac{D}{\ln N} = \lim_{N \to \infty} \frac{2(\sqrt{N} - 1)}{\ln N} = \infty$$

NO SWDP