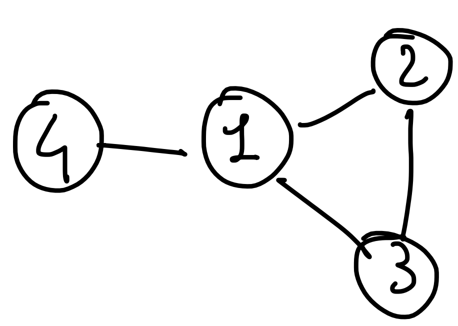
# WEEK 3 Lecture 2

## 2.6 SUBGRAPHS, CYCLES and CLIQUES

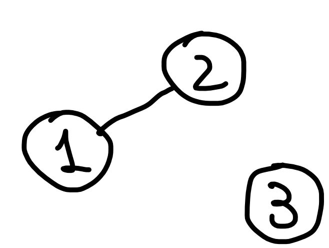
**DEF** SUBGRAPH

A subgraph $H=(V',E')$ of a network $G=(V,E)$ is the network formed by a set of nodes $V' \subset V$ and a set of links $E' \subset E$ such that all the links in $E'$ are incident to the nodes in $V'$

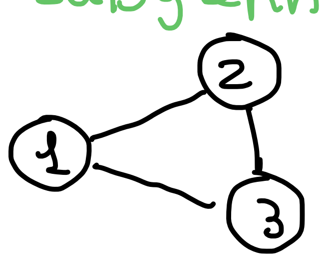
**EX** Network G



a subgraph of G



an induced subgraph



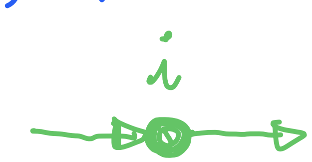
## DEF INDUCED SUBGRAPH

The induced subgraph $H=(V',E')$ of a network $G=(V,E)$ induced by the set of nodes $V'\subset V$ is the network formed by $V'$ and by the set $E'\subset E$ of all the links of $G$ incident to nodes in $V'$

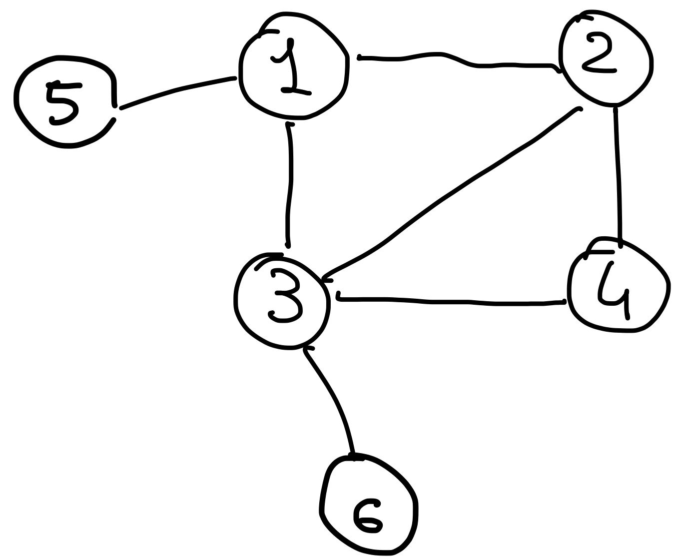
Particular types of subgraphs:

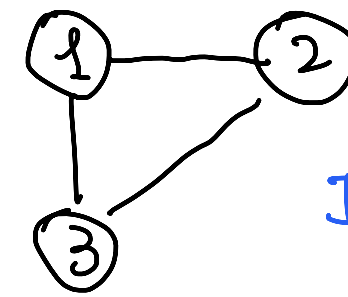
## DEF CYCLES

- An undirected cycle of $G$ is a subgraph $H=(V',E')$ of an undirected network $G=(V,E)$ such that
  1. every node $i\in V'$ has degree $k_i=2$
  2. every node can be reached by all other nodes
- A directed cycle of $G$ is a subgraph $H=(V',E')$ of a directed network $G=(V,E)$ such that
  1. every node $i\in V'$ has $k_i^{IN}=k_i^{OUT}=1$
  2. every node can be reached by all other nodes



Ex

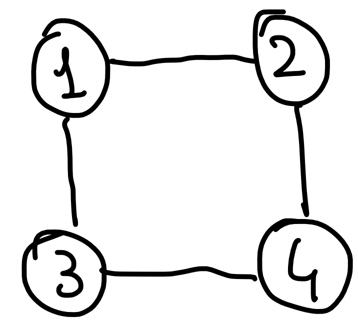



Cycle of length 3

TRIANGLE

In a cycle $|V'| = |E'|$ is the length of the cycle




Cycle of length 4

QUADRILATER

**THEO**

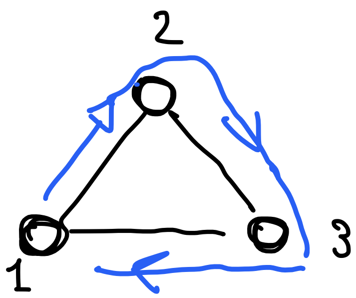
The # of undirected cycles of length 3 (TRIANGLES) in an undirected network is given by: $\frac{1}{6} \mathrm{Tr}(\mathbb{A}^3)$

The # of directed cycles of length 3 (DIRECTED TRIANGLES) in a directed network is $\frac{1}{3} \mathrm{Tr}(\mathbb{A}^3)$

Proof $\quad \mathrm{Tr}(\mathbb{A}^n) = $ # of closed walks of length $n$

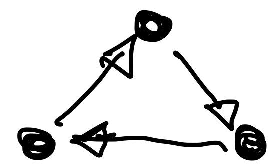
hence $\quad \mathrm{Tr}(\mathbb{A}^3) = $ # of cyclic paths of length 3

Now each triangle in $G$ corresponds to 6 cyclic paths




- We can start at each of the 3 nodes
- we can rotate clock- or anticlockwise

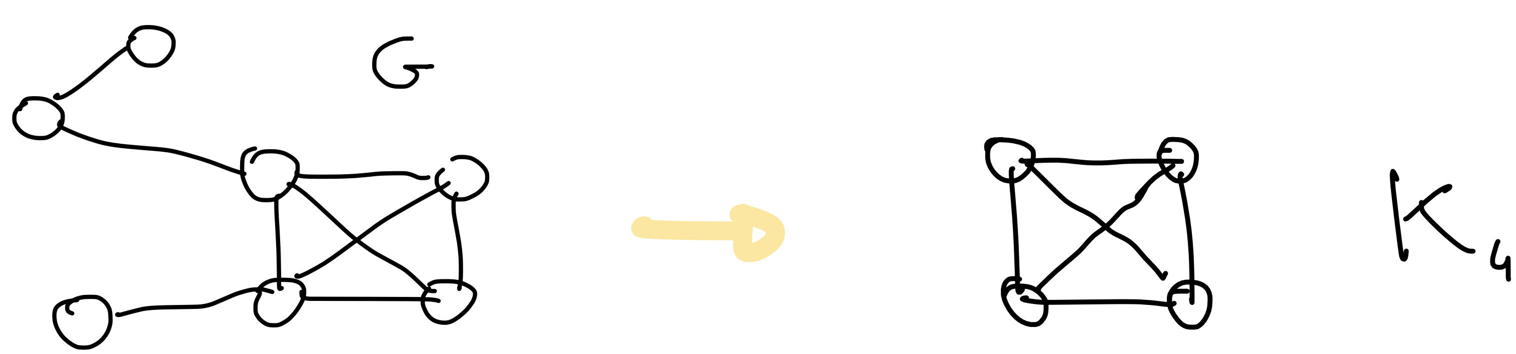
Instead each directed triangle corresponds to 3 directed cyclic paths



## DEF CLIQUES

A clique $K_n$ is a subgraph $G' = (V', E')$ with $|V'| = n$ of an undirected network $G = (V, E)$ such that every node is linked to every other node

**EX**



# 2.7 CONNECTEDNESS and COMPONENTS

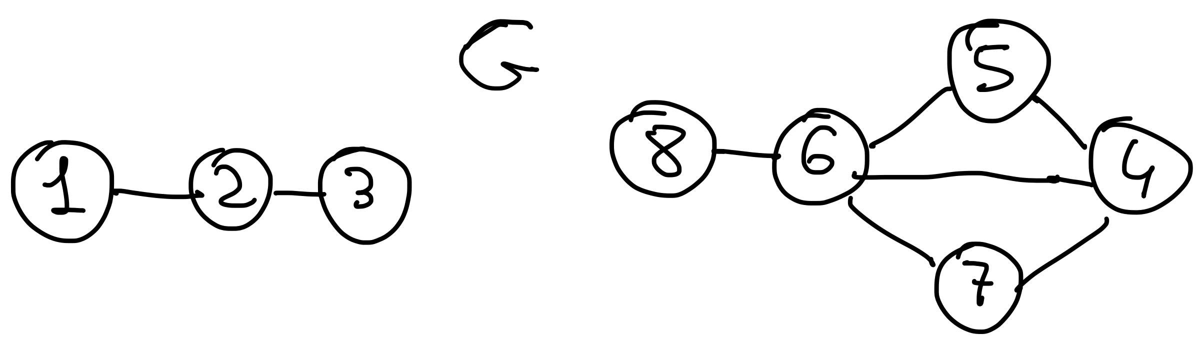
## Undirected networks

**DEF** CONNECTEDNESS

Two nodes $i$ and $j$ of an undirected network $G$ are CONNECTED if there exists a path between $i$ and $j$

$G$ is CONNECTED if all pairs of nodes are connected

**EX**




$N = 8$ nodes  unconnected network

e.g. 1 and 4 are not connected

However the network is made by 2 components

$\{1, 2, 3\}$  $\{4, 5, 6, 7, 8\}$

**DEF** COMPONENT

A component of G associated to node $i$ is the maximal connected induced subgraph containing $i$

the subgraph induced by all the nodes connected to $i$

## Directed networks

(A) We can either ignore the direction of links

**DEF** WEAKLY-CONNECTED COMPONENT (WCC)

A directed network $G=(V,E)$ is weakly connected if the underlying undirected graph $G^u$ is connected

the graph you get by neglecting the direction of the links

A WCC of $G$ is a component of $G^u$

**(B)** Or we can take the direction of links into account

**DEF** STRONG CONNECTEDNESS

Two nodes $i$ and $j$ of a directed network $G(V,E)$ are strongly connected if there is a path from $i$ to $j$ and a path from $j$ to $i$

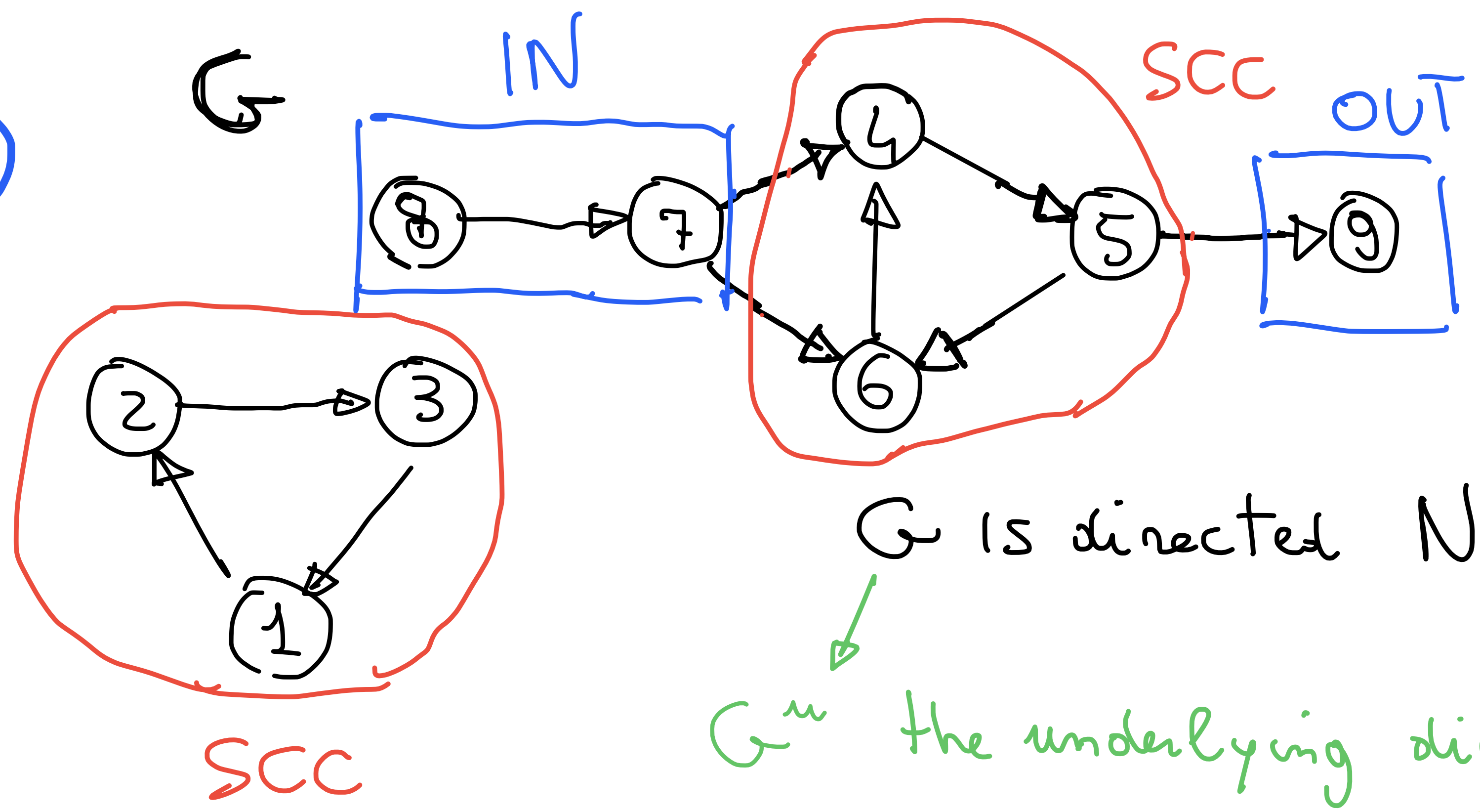
$G$ is said to be STRONGLY CONNECTED if all pairs of nodes are strongly connected

**DEF** STRONGLY-CONNECTED COMPONENT (SCC)

A strongly-connected component of a directed network $G(V,E)$ associated to node $i$ is the maximal strongly-connected induced subgraph containing $i$

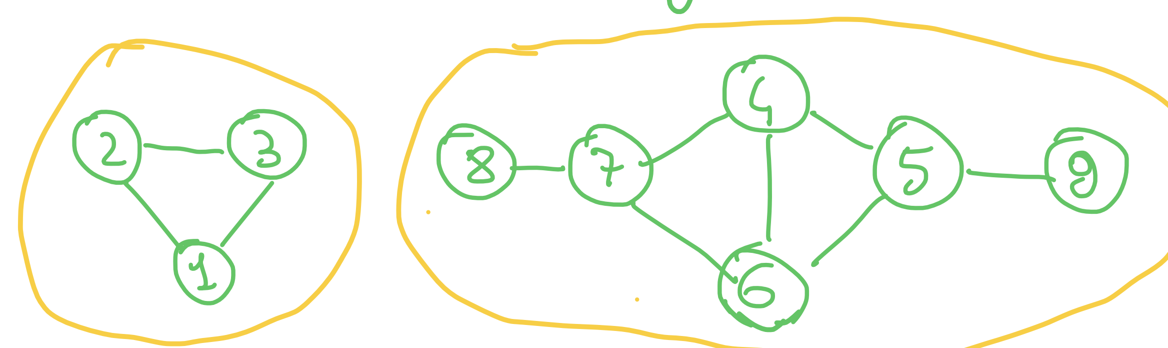
the subgraph induced by all nodes which are strongly connected to i

Ex




G is directed $N=9$ $L=10$

$G^u$ the underlying directed network




is not connected and has 2 components:

$\{1,2,3\}$ $\{4,5,6,7,8,9\}$

So G is NOT weakly connected

and has 2 WCCs : $\{1,2,3\}$, $\{4,5,6,7,8,9\}$

G is NOT strongly connected and has ~~2~~ 5 SCCs

$\{1, 2, 3\}$, $\{4, 5, 6\}$ $\{7\}$ $\{8\}$ $\{9\}$

Moreover given a SCC we can look at all nodes from which we can reach the SCC, and all the nodes that we can reach from the SCC

$\boxed{\text{DEF}}$ IN- and OUT-COMPONENT of a SCC

The in-component relative to a given SCC is the set of nodes from which there is a directed path to the SCC, but there is no directed path from the SCC to them

The out-component - - - -

analogous

Ex

# The Bow Tie of the World Wide Web