WEEK 3 Lecture 1

2.4 WALKS and PATHS

There are different ways we can visit a network

DEF WALKS, TRAILS and PATHS

A WALK from a node ⓘ to a node Ⓙ is a sequence of nodes (starting in i and ending in J) such that every consecutive pair of nodes is connected by a link. The LENGTH of the walk is the # of traversed links

A TRAIL is a walk in which no link is repeated

A PATH is a walk in which no node is repeated

visited more than once

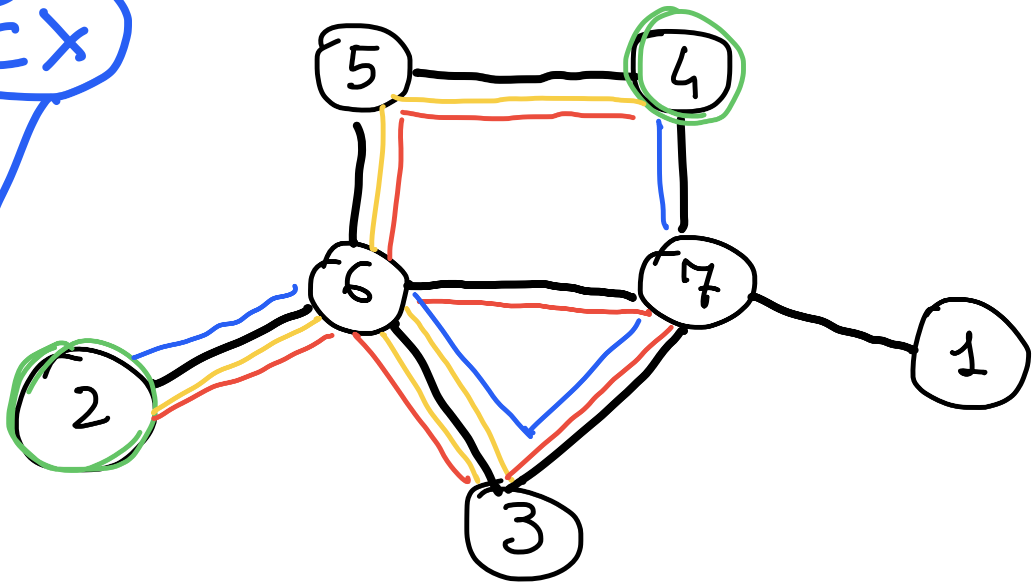
DEF CIRCUITS and CYCLIC PATHS

A CIRCUIT is a closed trail (i.e. a trail where the initial node i and the final node J coincide)

A CYCLIC PATH is a closed walk of at least 3 nodes where i and J coincide and all other nodes are distinct from each other and from i

"a closed path"

Ex



i = 2 J = 4

(2, 6, 3, 6, 5, 4) is a walk of length 5

(2, 6, 3, 7, 6, 5, 4) is a trail of length 6

nodes are repeated but links are not

no node is repeated

(2, 6, 3, 7, 4) is a path of length 4

→ $(\underline{2, 6, 5, 4})$ is a <u>path</u> of length 3

Also: $(⑥, 3, 7, ⑥)$ is a cyclic path of length 3

$(⑤, 4, 7, 6, ⑤)$ is a cyclic path of length 4

The same definitions apply also to directed networks

PROPOSITION NUMBER OF WALKS

In an unweighted (either undirected or directed) network, the # of walks (directed walks) N^n_{ij} of length n joining node j to node i is given by:

$$N^n_{ij} = [A^n]_{ij}$$

↖ indicates the entry i, j of matrix $A^n = \underbrace{A \cdot A \cdot \ldots A}_{n \text{ times}}$

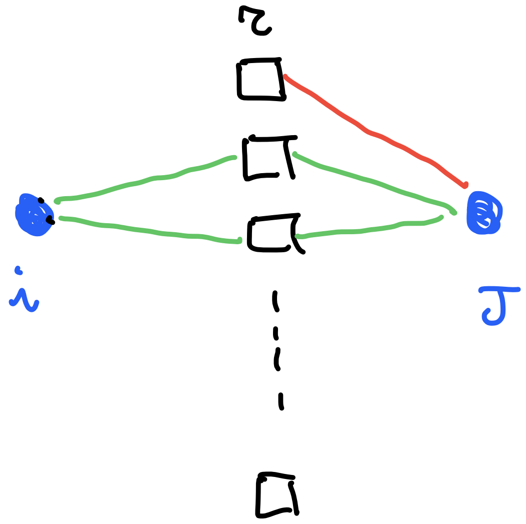
Proof

- For $n=1$ $$N^{1}_{iJ} = [A]_{iJ} = A_{iJ} = \begin{cases} 1 & \text{if } (J,i) \in E \\ 0 & \text{otherwise} \end{cases}$$

- For $n=2$ $$N^{2}_{iJ} = [A^2]_{iJ} = \sum_{r=1}^{N} \underbrace{A_{ir} A_{rJ}}$$

if (J,r,i) is a walk

$$A_{ir} A_{rJ} = \begin{cases} 1 & \text{if both links } (J,r) \text{ and } (r,i) \text{ exist} \\ 0 & \text{otherwise} \end{cases}$$



For a more formal proof by induction on n see my book (pag 295)

As a particular case if $J=i$ we get that $[A^n]_{ii}$ ← diagonal term of A^n is equal to the # of closed walks of length n from a node i to itself:

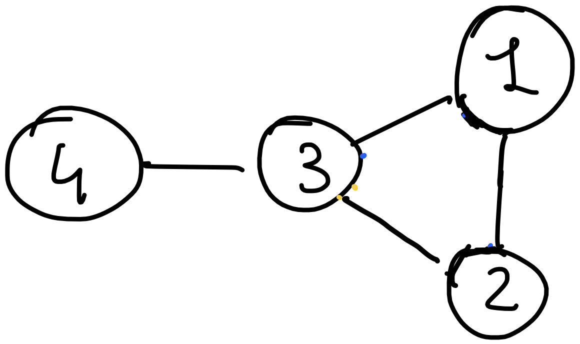
$$N^n_{ii} = \left[\mathbb{A}^n\right]_{ii}$$

Hence the TOTAL # of CLOSED WALKS of length n in a graph G is equal to

$$\sum_{i=1}^{N} N^n_{ii} = \mathrm{Tr}\left(\mathbb{A}^n\right)$$

trace

EX



$$\mathbb{A} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 4 & 0 & 0 & 1 & 0 \end{array}$$

symmetric

$N^1_{32} = 1$ (2,3)

$$\mathbb{A}^2 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

$N^2_{32} = 1$

(2, 1, 3)

1 walk of length 2 between nodes 2 and 3

$N^2_{33} = 3$ 3 closed walks from node 3 to itself $\begin{cases} (3,1,3) \\ (3,2,3) \\ (3,4,3) \end{cases}$

$$A^3 = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 3 & 2 & 4 & 1 \\ 4 & 4 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{pmatrix}$$

# of closed walks

$N^3_{11} = N^3_{22} = N^3_{33} = 2$

$N^3_{44} = 0 \leftarrow$ no ways to return to node 4 in 3 steps

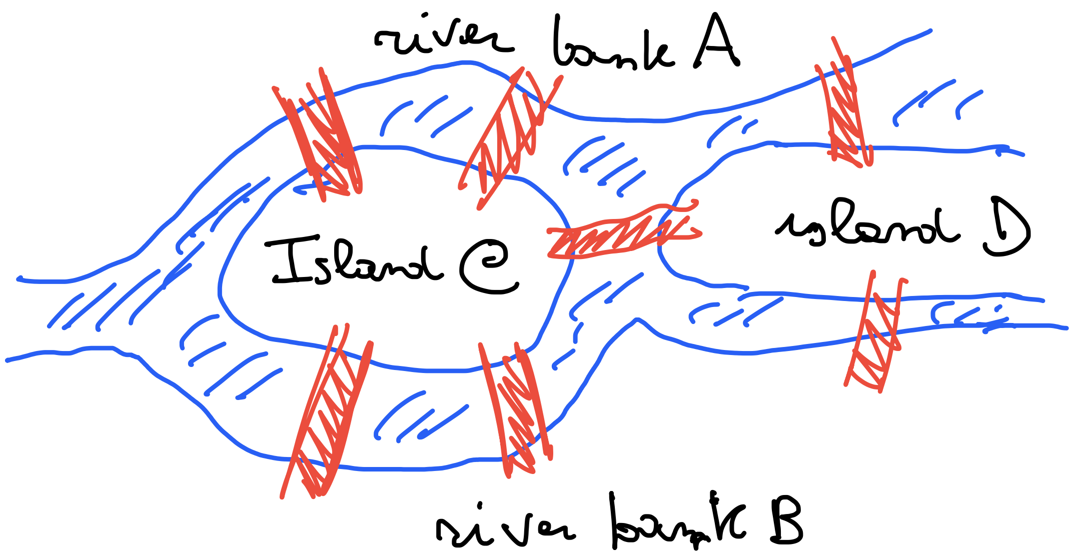
DEF EULERIAN CIRCUIT

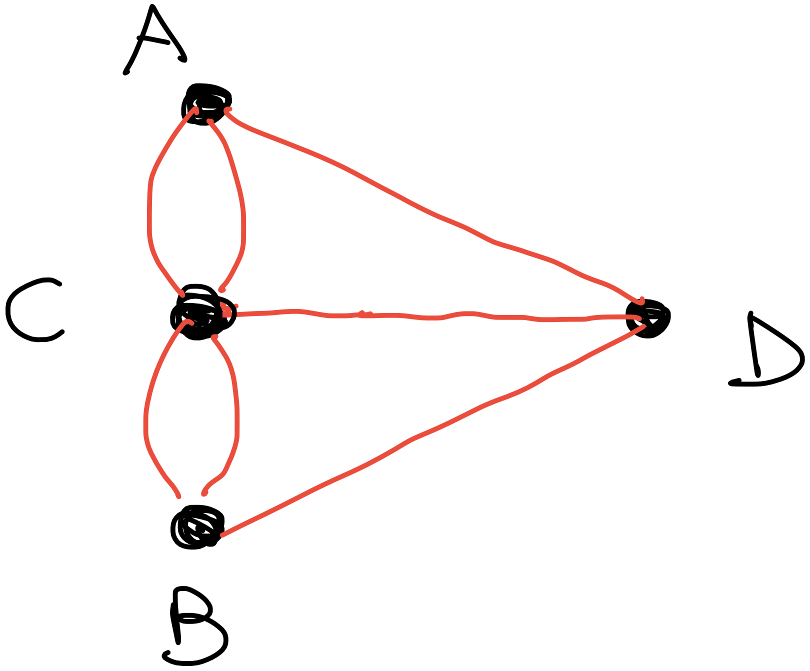
is a circuit that contains all the links in a network

a circuit is a closed trail

A network is said EULERIAN if it contains at least one Eulerian circuit

The problem of the 7 bridges of Königsburg





ORIGINAL QUESTION:

Is it possible to find a stroll that traverses edge of the bridges exactly once and returns to the starting point ?

→

GRAPH QUESTION

Is it possible to find an Eulerian circuit ?

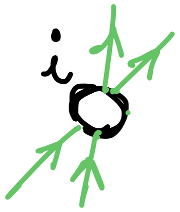
Solved by EULER in 1763

THEO EULER THEOREM

A connected graph is EULERIAN iff each node has EVEN degree

Proof: there is a Eulerian circuit ⇒ each node degree is even

Eulerian circuit ⇒ each node i is a crossing point



if p_i = # of times that we visit node i, then $k_i = 2p_i$, so k_i is EVEN

The converse was proved in 1873

ANSWER to the BRIDGE PROBLEM : **NO** !

because the graph is NOT EULERIAN

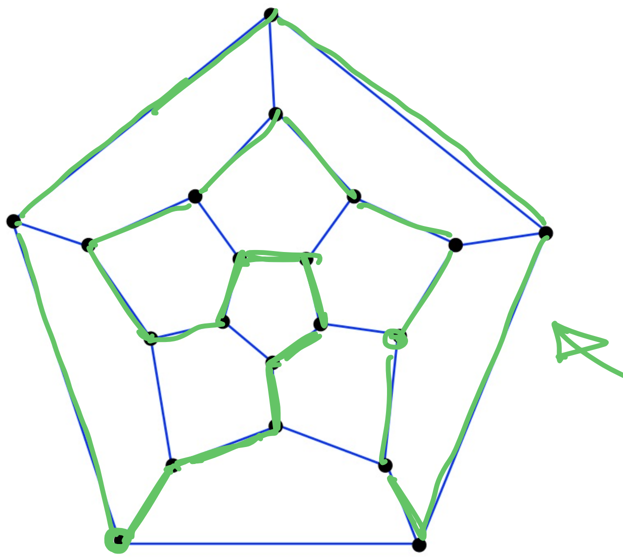
$k_A = k_B = k_D = 3$ $\quad k_C = 5$ ← NOT EVEN

DEF HAMILTONIAN CYCLE

A Hamiltonian cycle is a cyclic path that contains ALL the NODES of a network

A network is said HAMILTONIAN if it contains at least one Hamiltonian cycle

ES



Example of Hamiltonian cycle

2.5 DISTANCES

DEF **SHORTEST PATH**

A shortest path (or geodesic) between node J and node i is a walk of minimal length

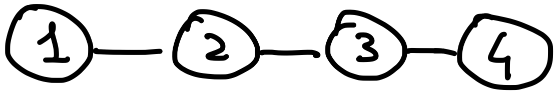
DEF **GRAPH DISTANCES**

The distance d_{iJ} between J and i is the length of a (any) shortest path between the two nodes

If there is No path between J and i we set $d_{iJ} = \infty$

EX Linear chain of N nodes

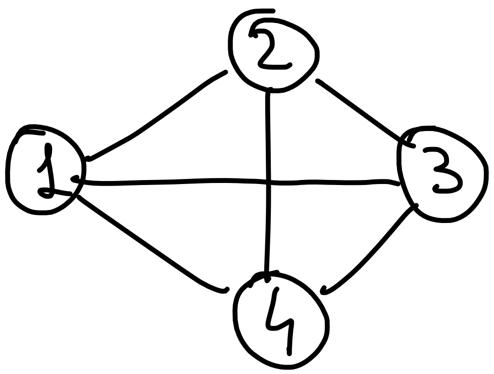
Take $N = 4$



$$d = \begin{array}{cccccc} 1 & 2 & 3 & 4 & \\ \left(\begin{matrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{matrix}\right. & & & & \left.\right) & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \end{array}$$

Complete network of N nodes

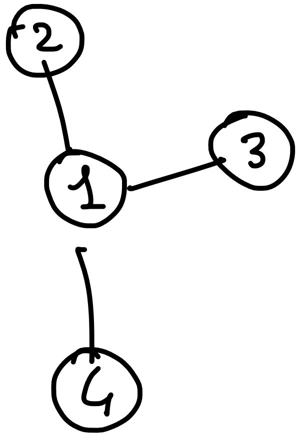
Take $N=4$



$$dll = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Anther Tree of N node

$N=4$



A tree is a connected network with no cyclic paths

N nodes $L = N-1$ links

$$dll = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 \\ 1 & 2 & 0 & 2 \\ 1 & 2 & 2 & 0 \end{pmatrix}$$

DEF AVERAGE DISTANCE (a.k.a. CHARACTERISTIC PATH LENGTH)

is the mean (shortest-path) distance over all pairs of nodes in the network

$$\ell = \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} d_{ij}$$

DEF DIAMETER

the diameter D of a network is $D = \max_{i,j} \{ d_{ij} \}$

By definition we have $D \geq \ell$

EX Linear chain, complete network, ...

See FA 2
on Friday

DEF SMALL-WORLD (DISTANCE) PROPERTY (SWDP)

A network has the SWDP if

$$D \simeq O(\ln N)$$ ← the diameter is of the same order of magnitude as ln N

big O

or if

$$D \simeq o(\ln N)$$ ← the diameter is of a smaller order of magnitude than ln N

small o

$$D \simeq O(\ln N) \Rightarrow \lim_{N \to \infty} \frac{D}{\ln N} = c < \infty$$

with $c > 0$

$$D \simeq o(\ln N) \Rightarrow \lim_{N \to \infty} \frac{D}{\ln N} = 0$$

Ex   See FA 2

IMPORTANT: Please submit your solution to

ASSESSED QUIZ 1

(deadline this afternoon at 5 PM)