WEEK 2 Lecture 2

DEF AVERAGE and MAXIMUM DEGREE

- In an undirected network

  the AVERAGE (NODE) DEGREE is $\boxed{\langle K \rangle = \frac{1}{N} \sum_{i=1}^{N} K_i}$

  while

  the MAXIMUM DEGREE is $\boxed{K = \max_{i=1,2,\dots,N} K_i}$

- In a directed network

  AVERAGE IN-DEGREE $\langle K^{IN} \rangle = \frac{1}{N} \sum_{i=1}^{N} K_i^{IN}$

  AVERAGE OUT-DEGREE $\langle K^{OUT} \rangle = \frac{1}{N} \sum_{i=1}^{N} K_i^{OUT}$

  MAXIMUM IN-DEGREE $K^{IN} = \max_i K_i^{IN}$

  MAXIMUM OUT-DEGREE $K^{OUT} = \max_i K_i^{OUT}$

PROPOSITION

The average degree $\langle k \rangle$ of a simple network satisfies:

$$\boxed{\langle k \rangle = \frac{2L}{N}}$$

Proof $\langle k \rangle = \frac{1}{N} \sum_i k_i = \frac{1}{N} \left(\sum_i \sum_j A_{ij} \right) = \frac{2L}{N}$

$k_i = \sum_j A_{ij}$

$L = \frac{1}{2} \sum_i \sum_j A_{ij}$

PROPOSITION

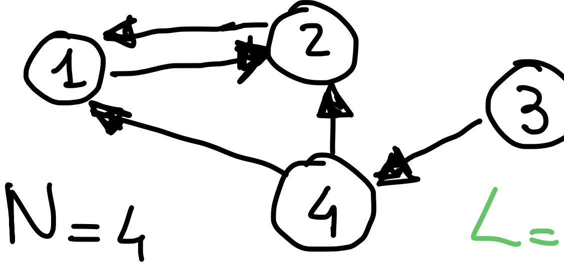
The average in- and out-degree of a directed network satisfy

$$\boxed{\langle k^{IN} \rangle = \langle k^{OUT} \rangle = \frac{L}{N}}$$

Proof $\langle k^{IN} \rangle = \frac{1}{N} \sum_i k_i^{IN} = \frac{1}{N} \left(\sum_i \sum_j A_{ij} \right) = \frac{L}{N}$

$\langle k^{OUT} \rangle = \frac{1}{N} \sum_i k_i^{OUT} = \frac{1}{N} \left(\sum_i \sum_j A_{ji} \right) = \frac{L}{N}$

EX Back to our previous example



$N=4$ $\quad L=5$

$$\langle k^{IN}\rangle = \frac{1}{4}\sum_{i=1}^{4} k_i^{IN} = \frac{1}{4}(2+2+0+1) = \frac{5}{4}$$

$\{2, 2, 0, 1\}$

$$K^{IN} = \max_i k_i^{IN} = 2$$

$$\langle k^{OUT}\rangle = \frac{1}{4}\sum_{i=1}^{4} k_i^{OUT} = \frac{1}{4}(1+1+1+2) = \frac{5}{4}$$

$\{1, 1, 1, 2\}$

$$K^{OUT} = \max_i k_i^{OUT} = 2$$

DEF DEGREE DISTRIBUTIONS

In an undirected the DEGREE DISTRIBUTION $P(k)$ is the fraction of nodes of degree k

$$\boxed{P(k) = \frac{N(k)}{N}}$$

$N(k)$ is the # of nodes of degree k

$$N(k) = \sum_{i=1}^{N} \delta(k_i, k)$$

It also indicates the probability that a randomly chosen node of the network has degree K

- In a directed network: the IN-DEGREE DISTRIBUTION $P^{IN}(k)$ is the fraction of nodes of in-degree equal to k

$$P^{IN}(k) = \frac{N^{IN}(k)}{N} \qquad N^{IN}(k) = \sum_{i=1}^{N} \delta\left(k_i^{IN}, k\right)$$

Analogously

$$P^{OUT}(k) = \frac{N^{OUT}(k)}{N} \qquad N^{OUT}(k) = \sum_{i=1}^{N} \delta\left(k_i^{OUT}, k\right)$$

Of course these distributions are defined for $0 \leq K \leq N-1$

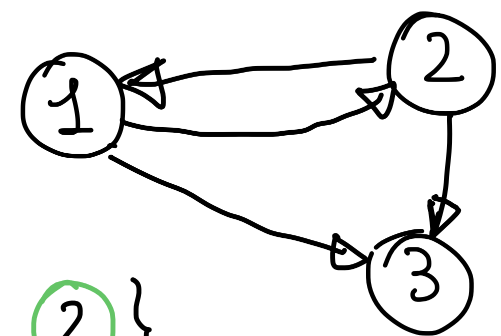
or N if we allow tadpoles

Ex

$$\mathbb{A} = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}$$

$\mathbb{A}$ is NOT symmetric
$\Downarrow$
the network is directed

$N = 3 \quad L = 4$



In-degree sequence $\{1, 1, 2\}$

Out-degree sequence $\{\boxed{2}, \boxed{2}, 0\}$

In-degree distribution $P^{IN}(k)$ defined $0 \leq k \leq 2$

$$P^{IN}(0) = \frac{N^{IN}(0)}{N} = \frac{0}{3} = 0 \qquad P^{IN}(1) = \frac{N^{IN}(1)}{N} = \frac{2}{3} \qquad P^{IN}(2) = \frac{N^{IN}(2)}{N} = \frac{1}{3}$$

$$P^{OUT}(0) = \frac{N^{OUT}(0)}{N} = \frac{1}{3} \qquad P^{OUT}(1) = \frac{N^{OUT}(1)}{N} = \frac{0}{3} = 0 \qquad P^{OUT}(2) = \frac{N^{OUT}(2)}{N} = \frac{2}{3}$$

You should always check that

$$\sum_k P^{IN}(k) = 1 \qquad \sum_k P^{OUT}(k) = 1$$

The average degree can be expressed in terms of the degree distribution

undirected networks $\langle k \rangle = \sum_k k P(k)$

Proof

$$\langle k \rangle = \sum_k k P(k) = \sum_k k \frac{N(k)}{N} = \frac{1}{N} \sum_k k \sum_{i=1}^{N} \delta(k_i, k) =$$

$$N(k) = \sum_{i=1}^{N} \delta(k_i, k)$$

$$= \frac{1}{N} \sum_{i=1}^{N} \underbrace{\sum_k k \, \delta(k_i, k)}_{k_i} = \underbrace{\frac{1}{N} \sum_{i=1}^{N} k_i}_{\text{which is the definition of } \langle k \rangle}$$

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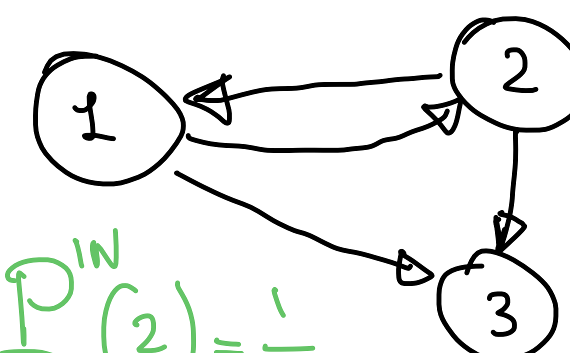
Analogously for directed networks

$$\boxed{\begin{aligned}\langle k^{IN}\rangle &= \sum_k k\, P^{IN}(k)\\ \langle k^{OUT}\rangle &= \sum_k k\, P^{OUT}(k)\end{aligned}}$$

EX Previous example



$P^{IN}(0) = 0 \qquad P^{IN}(1) = \frac{2}{3} \qquad P^{IN}(2) = \frac{1}{3}$

$$\langle k^{IN}\rangle = \sum_{k=0}^{2} k\, P^{IN}(k) = 0\cdot 0 + 1\cdot\frac{2}{3} + 2\cdot\frac{1}{3} = \frac{4}{3}$$ which is equal to

$$\langle k^{IN}\rangle = \frac{1}{N}\sum_{i=1}^{3} k_i^{IN} = \frac{1}{3}\left(1+1+2\right) = \frac{4}{3}$$

Similarly for $\langle k^{OUT}\rangle = \ldots$