WEEK 2 Lecture 1

From W1 → DEF → UNDIRECTED networks
→ SIMPLE networks
→ WEIGHTED networks

1.4 HOW TO REPRESENT a NETWORK

2 different methods → the EDGE LIST
→ the ADJACENCY MATRIX

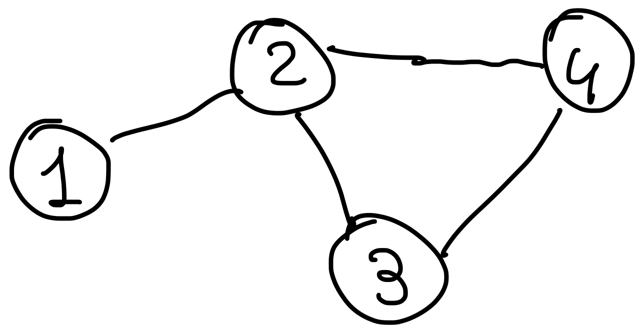
SIMPLE NETWORKS

DEF EDGE LIST of a SIMPLE NETWORK

is a list of the L pairs of node labels (j, i) indicating that there is a link between node i and node j

L is the total # of links in the network

Ex



$N = 4 \qquad L = 4$

Edge list

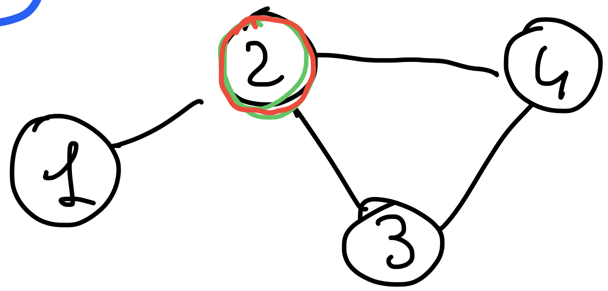
$$\{\overset{(2,1)}{(1,2)};\ (2,3);\ (2,4);\ (3,4)\}$$

every link occurs ONCE in the edge list of simple networks and $(i,j) = (j,i)$

DEF ADJACENCY MATRIX of A SIMPLE NETWORK

$G = (V, E)$ of $N = |V|$ nodes is a $N \times N$ square matrix A of elements
$$A_{ij} = \begin{cases} 1 & \text{if } (j,i) \in E \\ 0 & \text{otherwise} \end{cases}$$

Ex



$N = 4$

$$A = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

NO TADPOLES $\rightarrow$ all zeroes in the diagonal

$(j,i) = (i,j) \rightarrow A$ is a symmetric matrix

DIRECTED NETWORKS

A directed link starting from node j ("SOURCE") and ending at node i ("TARGET") is indicated by an ORDERED pair (j,i)

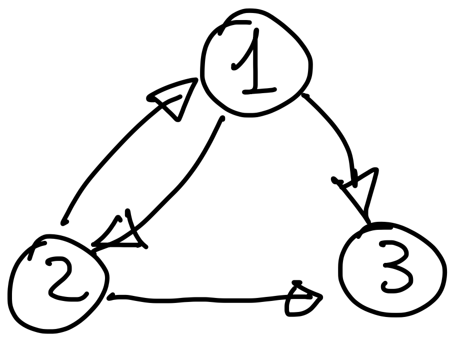
NOTE $(j,i) \neq (i,j)$

DEF EDGE LIST OF A DIRECTED NETWORK

is a list of L pairs of ordered node labels (j,i), each one indicating that node j points to node i

L is the total number of directed links in the network

Ex



$N = 3 \qquad L = 4$

Edge list is:

$$\{(1,2);\ (2,1);\ (1,3);\ (2,3)\}$$

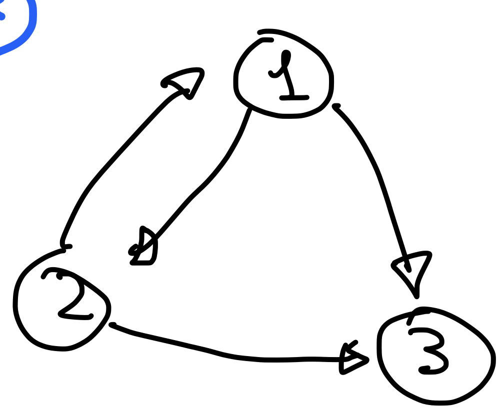
$\boxed{\text{DEF}}$ ADJACENCY MATRIX of a DIRECTED NETWORK

$G = (V, E)$ of $N = |V|$ nodes is a $N \times N$ square matrix $\mathbb{A}$ of elements

$$A_{ij} = \begin{cases} 1 & \text{if } (j, i) \in E \\ 0 & \text{otherwise} \end{cases}$$

if j POINTS to i

Ex



$$\mathbb{A} = \begin{array}{cc} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array}$$

NOT SYMMETRIC

WEIGHTED NETWORKS

Every link (either directed or undirected) is indicated by the triple (j, i, w_{ij}) where $w_{ij} > 0$ is the weight of the link (j, i)

DEF **EDGE LIST of a WEIGHTED NETWORK**
is a list of L triples (j, i, w_{ij})

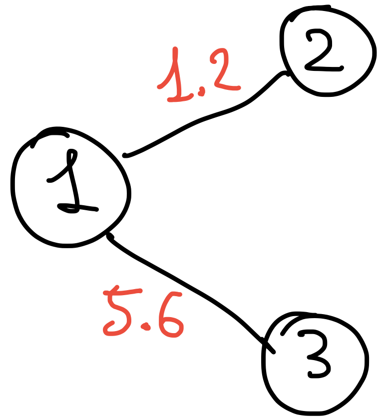
In an undirected network (j, i, w_{ij}) means

"j and i are linked with weight w_{ij}"

In a directed network (j, i, w_{ij}) means

"j points to i with weight w_{ij}"

EX



$N = 3 \quad L = 2$

Edge list is

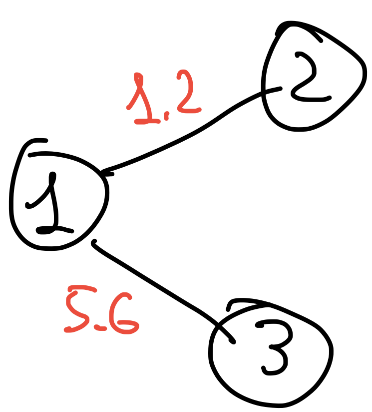
$$\{ (1, 2, 1.2) ; (1, 3, 5.6) \}$$

DEF ADJACENCY MATRIX of a WEIGHTED NETWORK

$G = (V, E)$ of $N = |V|$ nodes is a $N \times N$ square matrix

A of elements
$$A_{ij} = \begin{cases} w_{ij} & \text{if } (j, i) \in E \\ 0 & \text{otherwise} \end{cases}$$

EX



$$A = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 1.2 & 5.6 \\ 1.2 & 0 & 0 \\ 5.6 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}$$

TADPOLES

Are indicated by diagonal terms in the adjacency matrix $\mathbb{A}$

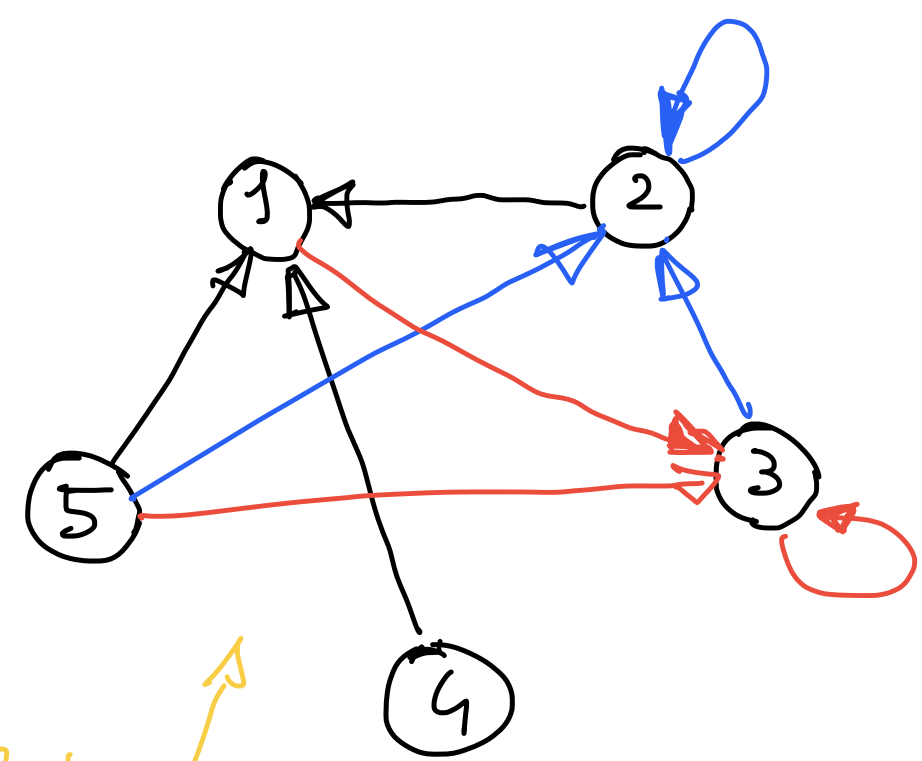
unweighted networks $\quad A_{ii} = \begin{cases} 1 & \text{if } (i,i) \in E \\ 0 & \text{otherwise} \end{cases}$

weighted networks $\quad A_{ii} = \begin{cases} w_{ii} & \text{if } (i,i) \in E \\ 0 & \text{otherwise} \end{cases}$

Ex

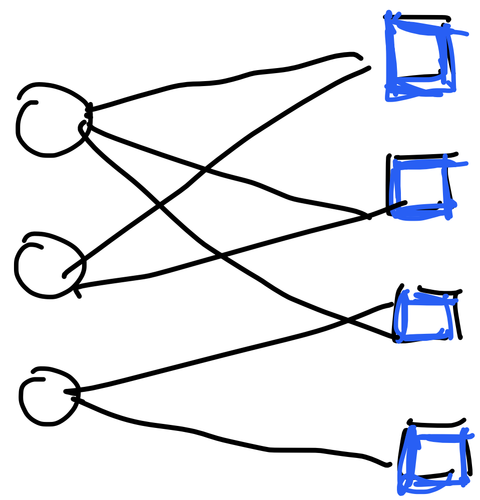
Check your understanding

$$\mathbb{A} = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$



keep adding links

1.5 BIPARTITE NETWORKS



users — items (books, movies)

DEF BIPARTITE NETWORK

$G \equiv G_B = (\mathcal{V}, \mathcal{U}, E)$ is a network formed by two non-overlapping sets of nodes $\mathcal{V}$ and $\mathcal{U}$, and by a set of links E, such that every link joins a node in $\mathcal{V}$ with a node in $\mathcal{U}$

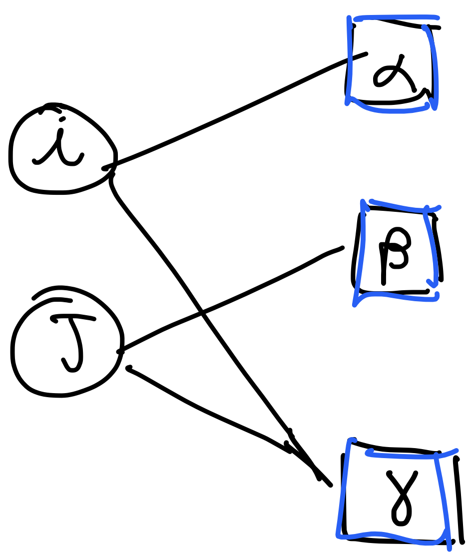
$N_{\mathcal{V}} = |\mathcal{V}| = \#$ of nodes in $\mathcal{V}$

$N_{\mathcal{U}} = |\mathcal{U}| = \#$ of nodes in $\mathcal{U}$

We indicate

$$\begin{cases} \text{a node in } V \text{ with Latin letters: } i, j, k, \dots \\ \text{a node in } U \text{ with Greek letters: } \alpha, \beta, \gamma \dots \\ \text{as } (i,\alpha) \text{ a link between node } i \in V \\ \qquad \text{and node } \alpha \in U \end{cases}$$

Ex



the edge list

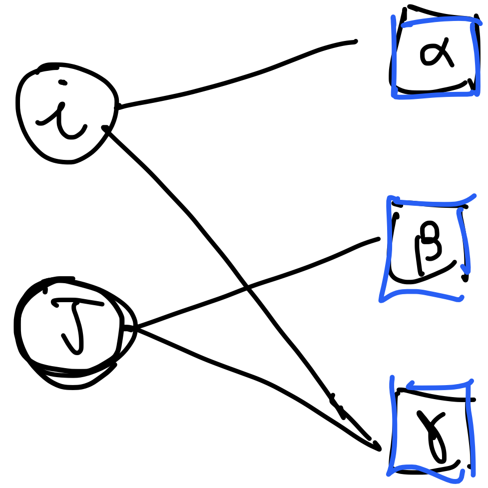
$$\{(i,\alpha); (i,\gamma); (j,\beta); (j,\gamma)\}$$

DEF INCIDENCE MATRIX of UNDIRECTED BIPARTITE NETWORKS

$G_B = (V, U, E)$ is a $N_V \times N_U$ matrix $\mathbb{B}$ of elements

$$B_{i\alpha} = \begin{cases} 1 & \text{if } (i,\alpha) \in E \\ 0 & \text{otherwise} \end{cases}$$

EX



$N_V = 2 \qquad N_U = 3$

$\mathbb{B}$ is a 2×3 matrix

$$\mathbb{B} = \begin{array}{c} \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} \alpha & \beta & \gamma \end{array} \\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \end{array}\begin{array}{c} \\ i \\ j \end{array}$$

- $\mathbb{B}$ can be easily extended to the case of weighted bipartite networks
- For directed bipartite networks we need two incidence matrices $\mathbb{B}$ and $\mathbb{B}'$

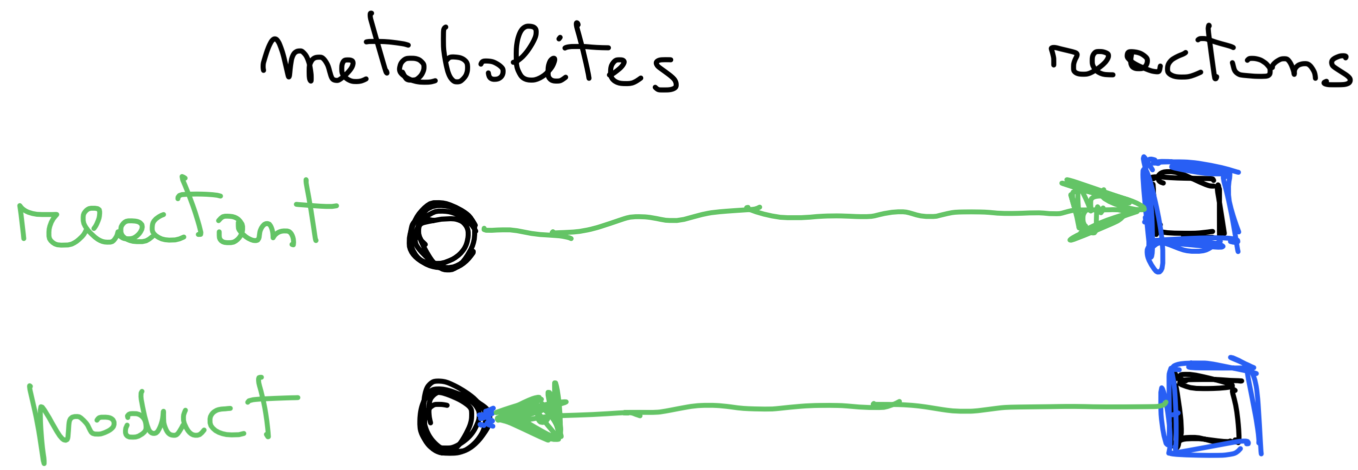
EX

METABOLIC NETWORKS

$V \equiv$ set of metabolites

$U =$ set of chemical reactions

Can be directed bipartite network if we adopt the following convention



CHAPTER 2
STRUCTURAL PROPERTIES

2.1 INTRODUCTION

$$\text{STRUCTURE} \longrightarrow \text{FUNCTION}$$

2.2 SIZE and NUMBER of LINKS

Size

N = total # of nodes

L = total # of links

Networks	Network size N
Brain	up to 10^{11}
Metabolic Networks	10^3
Social Networks	up to 10^9
Power-grids	up to 10^5
Internet	up to 10^5
WWW	10^9
Online social networks	10^8

Let us now express L as a function of $\mathbb{A}$

SIMPLE NETWORKS

$$L = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} A_{ij}$$

otherwise each link is counted twice

UNDIRECTED NETWORKS

$$L = \underbrace{\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} A_{ij}} + \frac{1}{2} \sum_{i=1}^{N} A_{ii}$$

here each tadpole is counted $\frac{1}{2}$

so we have to add this

DIRECTED NETWORKS

$$L = \sum_{i=1}^{N} \sum_{j=1}^{N} A_{ij}$$

2.3 DEGREE SEQUENCE and DISTRIBUTION

DEF NODE DEGREE

- The DEGREE k_i of node i in an undirected network is the # of links incident in i

$$k_i = \sum_{j=1}^{N} A_{ij} = \sum_{j=1}^{N} A_{ji}$$

- In a directed network

IN-DEGREE k_i^{IN} is the # of nodes pointing to node i

$$k_i^{IN} = \sum_{j=1}^{N} A_{ij}$$

OUT-DEGREE k_i^{OUT} is the # of nodes to which node i is pointing to

$$k_i^{OUT} = \sum_{j=1}^{N} A_{ji}$$

In a simple network $0 \leq K_i \leq N-1 \quad \forall i \in \{1, 2, ..., N\}$

$\boxed{\text{DEF}}$ DEGREE SEQUENCE

- In an undirected network the DEGREE SEQUENCE is the ordered set of the degrees of the nodes

$$\{K_i\}_{i=1,...,N} = \{K_1, K_2, ..., K_N\}$$

- In a directed network

IN-DEGREE SEQUENCE $\{K_i^{IN}\}_{i=1,...,N} = \{K_1^{IN}, K_2^{IN}, ..., K_N^{IN}\}$

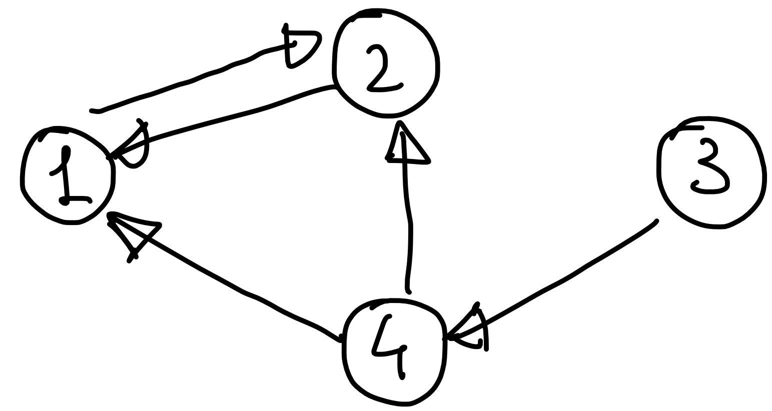
OUT-DEGREE SEQUENCE $\{K_i^{OUT}\}_{i=1,...,N} = \{K_1^{OUT}, K_2^{OUT}, ..., K_N^{OUT}\}$

Ex

$$\mathbb{A} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$N = 4$ nodes   directed network

$L = 5$ links



by looking at rows

In-degree sequence $\{2, 2, 0, 1\}$ $2+2+0+1 = 5 = L$

by looking at columns

Out-degree sequence $\{1, 1, 1, 2\}$ $1+1+1+2 = 5 = L$

Notice

$$\sum_{i=1}^{N} k_i^{IN} = \sum_{i=1}^{N} \left(\sum_{j=1}^{N} A_{ij} \right) = L$$

$$k_i^{IN} = \sum_{j=1}^{N} A_{ij}$$

$$\sum_{i=1}^{N} k_i^{OUT} = \sum_{i=1}^{N} \underbrace{\left(\sum_{j=1}^{N} A_{ji} \right)}_{k_i^{OUT}} = L$$