<u>Theorem 3.5</u> Let n be a positive integer. If n^2 is even then n is even.

Direct attack: Suppose n^2 is even. Then what?

<u>Proof</u> Prove the contrapositive: "if n is odd then n^2 is odd".
Write $n = 2k+1$ for some integer k. Then $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd. $\boxtimes$

Recall: a number x is rational if $x = a/b$ where a, b are integers and $b \neq 0$. x is irrational if it is not rational.

<u>Theorem 3.6</u> Let x be a positive real number. If x is irrational then $\sqrt{x}$ is irrational.

Direct attack: Suppose x is irrational, then what?

<u>Proof</u> Look at the contrapositive: if $\sqrt{x}$ is rational then x is rational. Write $\sqrt{x} = a/b$ with a, b integers and $b \neq 0$.

Then $x = (\sqrt{x})^2 = (a/b)^2 = a^2/b^2$, which is rational.

3.6 Special proof technique 2: Proof by contradiction.

We take the view that mathematical statements are either true or false. Then, to prove a statement P, we assert (not P) and derive a contradiction, such as $1=0$ or $(x<0)$ and $(x>1)$. Then (not P) is false and P is true.

Theorem 3.7 There is no rational number x such that $x^2=2$.

Recall that a/b is in lowest terms if a, b don't have a common factor.

Proof Use proof by contradiction. Suppose, for a contradiction, that there exists x with x rational and $x^2=2$. Write $x=a/b$ with a, b integers, $b \neq 0$ and a/b in lowest terms. Then $x^2 = (a/b)^2 = a^2/b^2 = 2$, and $a^2 = 2b^2$. $\circledast$

Then a^2 is even, and a is even by Theorem 3.5. Write $a=2k$ for some integer k. Substituting in $\circledast$ we get $(2k)^2 = 2b^2$, and hence $2k^2 = b^2$. By Theorem 3.5 we deduce that

b is even. The fact that a and b are both even contradicts the asscumption that a/b is in lowest terms. $\boxtimes$

Theorem 3.8 There is no smallest positive rational number.

Proof Assume, for a contradiction that x is the smallest positive rational number. Write x as $x = a/b$ with a, b integer, $b \neq 0$. Consider $y = a/2b$. Then y is rational, $y > 0$ and finally $y < x$ (since $x - y = a/b - a/2b = a/2b > 0$). This contracts the assumption that x is the smallest such. $\boxtimes$