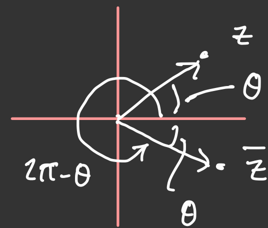
<u>Recap</u> Complex numbers $\mathbb{C}$ have the form $z = a+bi$ with $a, b \in \mathbb{R}$. $a = \mathrm{Re}(z)$ and $b = \mathrm{Im}(z)$. Arithmetic operations can be defined, and these have the usual properties. Cannot define a sensible ordering on $\mathbb{C}$, though. The complex plane and polar form $z = r(\cos\theta + i\sin\theta)$. r is the modulus $|z|$ of z and θ is the argument $\arg z$. To multiply two complex numbers, multiply the moduli and add the arguments.

- $|i| = 1$ and $\arg(i) = \frac{\pi}{2}$. So multiplication by i is the same as rotation (anticlockwise) by $\pi/2$



- $|\bar{z}| = |z|$ and $\arg \bar{z} = 2\pi - \arg z$. So $|z\bar{z}| = 1$ and $\arg z\bar{z} = \arg z + 2\pi - \arg z$
$$= 2\pi \equiv 0.$$
So $z\bar{z} \in \mathbb{R}$ and $z\bar{z} \geqslant 0$.

Since raising to the power $n \in \mathbb{N}$ is just repeated multiplication:

<u>Theorem 9.1</u> (De Moivre's Theorem) For all $n \in \mathbb{N}$ and all $\theta \in \mathbb{R}$

we have $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$.

Proof Let $z = \cos\theta + i\sin\theta$. Then $|z| = 1$ and $\arg z = \theta$.
Thus, $|z^n| = |z|^n = 1^n = 1$, and $\arg(z^n) = n\arg z = n\theta$. $\boxtimes$

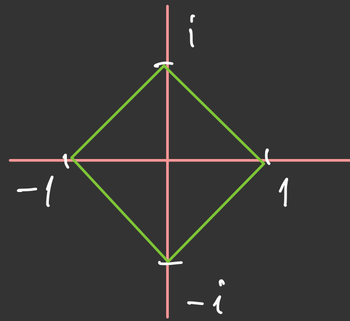
Question: What are the solutions to the equation $z^n = 1$ (so-called "roots of unity")? In $\mathbb{R}$ the solutions are $z = 1$ when n is odd and $z = \pm 1$ when n is even. In $\mathbb{C}$ we get many more solutions.

Let $z = r(\cos\theta + i\sin\theta)$. Then $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$.
For $z^n = 1$ we need $r^n(\cos(n\theta) + i\sin(n\theta)) = 1\cdot(\cos 0 + i\sin 0)$.
Thus $r = 1$ and $n\theta = 2m\pi$ for some $m \in \mathbb{N}$. So the argument θ must be of the form $\theta = \frac{2m\pi}{n}$.
In summary $z = \cos\frac{2m\pi}{n} + i\sin\frac{2m\pi}{n}$ for $m = 0, 1, 2, \ldots, n-1$.

Examples. Solutions to $z^4 = 1$?

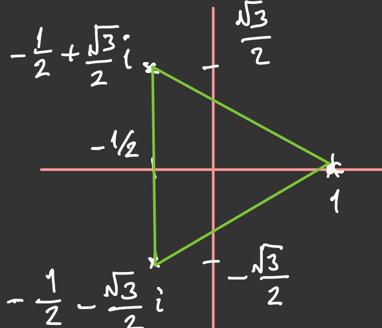
- $m = 0$: $\cos 0 + i\sin 0 = 1$
- $m = 1$: $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

- $m=2$: $\cos\pi + i\sin\pi = -1$
- $m=3$: $\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i$.
- [$m=4$: $z=1$.]



Solutions to $z^3 = 1$?

- $m=0$: $z = \cos 0 + i\sin 0 = 1$
- $m=1$ $z = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$
- $m=2$ $z = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$
- [$m=3$: $z = \cos 2\pi + i\sin 2\pi = 1$]



We deduce that $z^n = 1$ has exactly n solutions in complex numbers.

Fundamental Theorem of Algebra.

Every polynomial equation of degree n (i.e., $p(z) =$

$a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0 = 0$) has exactly n roots in $\mathbb{C}$, counted according to multiplicity; that is,

$$p(z) = (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n) = 0,$$

for some $\alpha_1, \alpha_2 \dots \alpha_n \in \mathbb{C}$.

We started off by adding to our number system the solution to a <u>particular</u> equation $z^2 + 1 = 0$, and find that we can now solve every polynomial equation!