Recap: Upper bound for X. Supremum (least upper bound) of X; supremum, if it exists, is unique. The supremum of X may exist even when X does not have a maximum. $\{x \in \mathbb{Q} : x^2 \leq 2\}$ has $\sqrt{2}$ as its supremum. Every set $X \subseteq \mathbb{R}$ that is bounded above has a supremum. As with upper bounds, so with lower: infimum.

9.1 Complex numbers – Definition

So far $\mathbb{N} \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{R}$.

Still cannot "solve" the equation $x^2 + x + 1 = 0$. To deal with this we introduce a number i which solves $x^2 + 1 = 0$. So $i^2 = -1$. Now see where this takes us.

Definition: A complex number z is an expression of the form $z = a + bi$. The set of all complex numbers is denoted $\mathbb{C}$.

Write $a = \mathrm{Re}(z)$ and $b = \mathrm{Im}(z)$. We can think of $\mathbb{R}$ as being a subset of $\mathbb{C}$: $\mathbb{R} = \{z \in \mathbb{C} : \mathrm{Im}(z) = 0\}$.

We next define the arithmetic operations $+, -, \times, \div$.

- Addition. $(a+bi)+(c+di) = (a+c)+(b+d)i$.
- Subtraction. $(a+bi)-(c+di) = (a-c)+(b-d)i$.
- Multiplication. $(a+bi)(c+di) = (ac-bd)+(ad+bc)i$

<u>Definition</u> The complex conjugate of $z = a+bi$ is $\bar{z} = a-bi$.

Note that $z\bar{z} = (a+bi)(a-bi) = a^2+b^2 \in \mathbb{R}$. Note also that $z\bar{z} \geqslant 0$ and $z\bar{z} = 0 \Rightarrow z = 0$

- Division $\dfrac{a+bi}{c+di} = \dfrac{a+bi}{c+di}\,\dfrac{c-di}{c-di} = \dfrac{(ac+bd)+(-ad+bc)i}{c^2+d^2}$

$$= \frac{ac+bd}{c^2+d^2} + \frac{(bc-ad)}{c^2+d^2}\,i.$$ Division is defined provided

$c \neq 0$ or $d \neq 0$, i.e., $z \neq 0$.

<u>Examples</u>
- $(2+3i)+(4-i) = 6+2i$.
- $(2+3i)-(4-i) = -2+4i$.
- $(2+3i)(4-i) = (8+3)+(-2i+12i) = 11+10i$.
- $(2+3i)/(4-i) = (2+3i)(4+i)/(4-i)(4+i)$

$$= [(8-3)+(2+12)i]/(16+1)$$

$$= \frac{5}{17} + \frac{14}{17} i.$$

However we cannot define an ordering of $\mathbb{C}$ that is consistent with the arithmetic operations. Illustration: What is the relationship between 0 and i?

- If $i > 0$ then $i^2 > 0$ which means $-1 > 0$
- If $i < 0$ then $(-i)^2 > 0$ which means $-1 > 0$

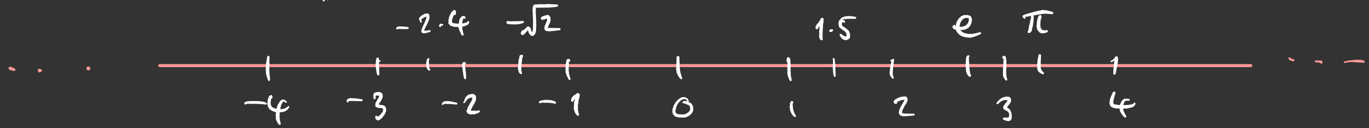
Returning to $x^2 + x + 1 = 0$, we can write down the solution:

$$x = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{-3}}{2} = -\frac{1}{2} \pm \frac{\sqrt{-1 \times 3}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i.$$

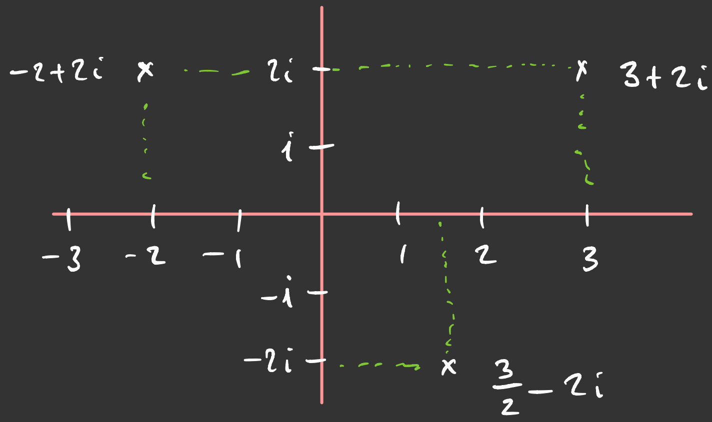
We can do the same thing with an quadratic equation.

9.2 The complex plane

We can visualise the real numbers $\mathbb{R}$ as a line:



We can't do this for the complex numbers because there is no natural order. But we can plot complex numbers in $\mathbb{R}^2$, with the point (a,b) corresponding to $a+bi$.



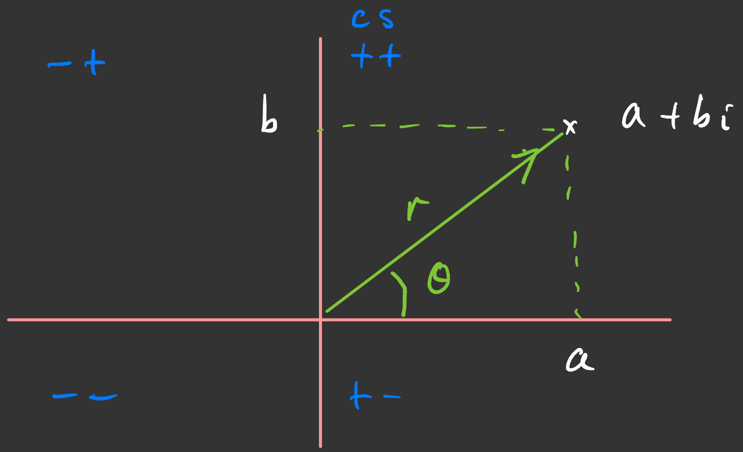
Quiz. Find a bijection $\mathbb{R}^2 \to \mathbb{R}$ (i.e., from $\mathbb{C}$ to $\mathbb{R}$).

$x = .x_1x_2x_3\cdots$ $\quad y = .y_1y_2y_3\cdots$ Define $f: \mathbb{R}^2 \to \mathbb{R}$ so that

$f(x,y) = 0.x_1y_1x_2y_2x_3y_3\cdots$ Check this has an inverse.

Given $f(x,y)$ we can deinterleave the odd digits $0.x_1x_2x_2\cdots$ to get x, and similarly for y.

Diagram above ($\mathbb{R}^2$) suggests looking at complex numbers in polar form.



Mapping from polar to cartesian is straightforward:

$$\mathbb{R} \times [0, 2\pi) \to \mathbb{R} \times \mathbb{R}, \qquad (r, \theta) \mapsto (r\cos\theta, r\sin\theta)$$

$$= r(\cos\theta + i\sin\theta).$$

This function is invertible.

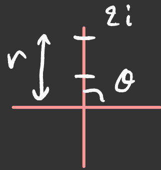
Given (a, b) or $a + ib$ what are r an θ ?

- $r = \sqrt{a^2 + b^2} \quad \left(= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = \sqrt{r^2} = r \right)$
- θ is the unique angle θ such that $r\cos\theta = a$

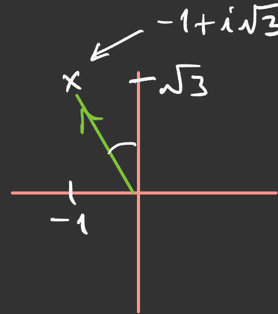
and $r\sin\theta = b$. I.e. $\cos\theta = \frac{a}{r} = \frac{a}{|z|}$, where $z = a+ib$,
and $\sin\theta = \frac{b}{r} = \frac{b}{|z|}$.

r is the modulus of z denoted $|z|$, and θ is the argument of z denoted $\arg z$.

(It is almost the case that $\theta = \arctan(b/a)$, but this only works when $\mathrm{Re}(z) = a > 0$.)



<u>Examples</u> • $z = 2i$. Then $r = 2$, $\theta = \frac{\pi}{2}$.

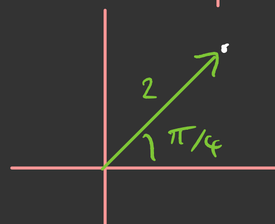


• $z = -1+\sqrt{3}i$. Then $r = 2$, $\theta = \frac{2\pi}{3}$.

(Looking at $\cos\theta$ gives $\theta \in \{\frac{2\pi}{3}, \frac{4\pi}{3}\}$

" " $\sin\theta$ gives $\theta \in \{\frac{\pi}{3}, \frac{2\pi}{3}\}$.)

• $|z| = 2$ $\arg z = \pi/4$. Then $z = \sqrt{2} + i\sqrt{2}$.



• How does addition look in $\mathbb{R}^2$? It looks like addition of vectors.

- How does multiplication look? Take $z = r(\cos\theta + i\sin\theta)$ and $w = s(\cos\varphi + i\sin\varphi)$. Then

$$zw = rs\left[(\cos\theta\cos\varphi - \sin\theta\sin\varphi) + (\cos\theta\sin\varphi + \sin\theta\cos\varphi)i\right]$$
$$= rs\left[\cos(\theta+\varphi) + i\sin(\theta+\varphi)\right].$$

So $|zw| = rs = |z| \times |w|$

$\arg(zw) = \theta + \varphi = \arg z + \arg w$.

<u>Examples</u>

$$(1+i)\times(-1-i) = \sqrt{2}\left(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4}\right)\sqrt{2}\left(\cos\tfrac{5\pi}{4} + i\sin\tfrac{5\pi}{4}\right)$$
$$-2i = 2\left(\cos\tfrac{3\pi}{2} + i\sin\tfrac{3\pi}{2}\right)$$

✓