<u>Recap</u> • Rational numbers $\mathbb{Q}$: definition as fractions a/b with $a, b \in \mathbb{Z}$ and $b \neq 0$. Similarities and differences to the integers.

- Real numbers $\mathbb{R}$: definition in terms of (possibly infinite) decimal expansions; $\mathbb{Q}$ is the subset of $\mathbb{R}$ with eventually periodic expansions.
- Maximum of a set of numbers; upper bound and bounded above.

Examples: • $\{x \in \mathbb{Q} : 0 < x < 1\}$ is bounded above but does not have a maximum, (*)

- $\emptyset$ is bounded above but does not have a maximum.

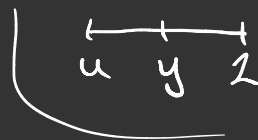
<u>Definition</u> Let X be a non-empty set of real numbers. We say that s is a supremum (= least upper bound) if s is an upper bound for X, and $t \geq s$ for any upper bound of X.

Idea is to deal with (*), which "ought" to have a maximum but don't.

<u>Examples</u> • Any finite set has a supremum (just take the

largest element in the set.

- If the set X has a maximum m, then m is also a supremum of X. (For all $x \in X$, we have $x \leq m$, since m is the maximum; clearly there is no smaller upper bound than m.)
- $X = \{x \in \mathbb{R} : 1 < x < 2\}$ has 2 as a supremum. Clearly, 2 is an upper bound. If $u \in \mathbb{R}$ is a smaller upper bound, then certainly $u \geq 3/2$; consider $y = \frac{u+2}{2}$ and observe that $1 < u < y < 2$ which contradicts u being an upper bound.



- $\emptyset$ is bounded above, but does not have a supremum. ($0, -1, -2, -3, \ldots$ are all upper bounds, but there is no least one.)

<u>Lemma 8.1</u> If $X \subseteq \mathbb{R}$ has a supremum then that supremum is unique.

Proof Suppose that s and t are both supremums of X. Then $s \leq t$ since s is a least upper bound and t is just some upper bound for X. Symmetrically $t \leq s$. $\boxtimes$

We talk about <u>the</u> supremum of X and write $\sup X$. By convention, $\sup X = \infty$ if X is not bounded above and $\sup \emptyset = -\infty$.

Informally, $\sup X$ acts like $\max X$ when $\max X$ does not exist.

Back to $\mathbb{R}$: Let $A = \{x \in \mathbb{Q} : x^2 \leq 2\}$. We hope that $\sup A = \sqrt{2}$.
- Let $x \in A$ be arbitrary. Then $x^2 \leq 2$ and so $x \leq \sqrt{2}$. I.e, $\sqrt{2}$ is an upper bound. For a contradiction, suppose u is a smaller upper bound on A. Choose a rational $q \in \mathbb{Q}$ with $u < q < \sqrt{2}$ (+), and $q \geq 0$. Then $q^2 \leq 2$ and so $q \in A$. Thus q is bigger than the alleged upper bound u. $\#$

(+) See Week 12 tutorial sheet!

Remark. A does not have a maximum. (+)

What happened above with the A happens in general.

<u>Theorem 8.2</u> If X is a non-empty subset of $\mathbb{R}$ that is bounded above, then X has a supremum.

<u>Proof</u> Outside of the module.

Everything we did with upper bounds can be done with lower bounds, in which case supremum becomes infimum.