<u>Recap</u> • Special kinds of relations: partial orders (reflexive, antisymmetric, transitive; equivalence relations (reflexive, symmetric, transitive).
• Sequences and subsequences; [weakly] increasing, [weakly] decreasing, constant.

<u>Example</u> The sequence 3, 1, 4, 1, 5, 9, 2, 6, 5, ... of digits of π.
Does there exist an increasing subsequence? ✗ decreasing subsequence? ✗ constant subsequence? ✓
"Eventually" can be applied to any of these properties: eventually increasing means (in the context of $(a_k)_{k=1}^{\infty}$) that there exists $n \in \mathbb{N}$ such that $a_k < a_{k+1}$ for all $k \geq n$. An example is $((k-4)^2)_{k=1}^{\infty} = 9, 4, 1, 0, 1, 4, 9, 16, \ldots$ which is eventually increasing (after $k=4$).

<u>8.1 Rational numbers</u> We started with $\mathbb{N}$, and introduced $\mathbb{Z}$ in order to be able to do subtraction always. Now we introduce rational numbers $\mathbb{Q}$ so we can always do division (except by 0). After that we can "solve" the equation $5x = 2$.

<u>Definition</u> A rational number is a fraction a/b where $a, b \in \mathbb{Z}$ and $b \neq 0$.

We can then define $+, \times, -, \div, =, \leq$ on $\mathbb{Q}$ in the familiar way.
e.g. $\frac{a}{b} + \frac{c}{d} = \frac{ad+cb}{bd}$; and $\frac{a}{b} = \frac{c}{d}$ if $ad = bc$. (More pedantically, we could introduce an equivalence relation $\equiv$ where $\frac{a}{b} \equiv \frac{c}{d}$ iff $ad = bc$. Then the equivalence classes contain fractions which represent the same rational number.) We can check that the operations so defined satisfy the usual identities (commutativity, associativity, etc.).

Why do we not allow b to be zero? For example lets call $1/0 = \infty$ (infinity). Then $1 = 0 \times \infty$. No consider $2 \times 0 \times \infty$. We have

$$1 = 0 \times \infty = (2 \times 0) \times \infty = 2 \times (0 \times \infty) = 2 \times 1 = 2 \quad ⚡$$

Every time you extend the number system you lose something.

- Subsets of the positive rationals do not necessarily have a minimum element, e.g., $\{x \in \mathbb{Q} : 1 < x < 2\}$.
- For every pair $a, b \in \mathbb{Q}$ with $a < b$ there is an $x \in \mathbb{Q}$ such that $a < x < b$. No adjacent rationals, rationals are dense.

Can't do induction on $\mathbb{Q}^+ = \{q \in \mathbb{Q} : q \geq 0\}$!

base case

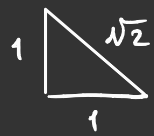
$n \in \mathbb{N}$: $P(n) \Leftarrow P(n-1) \Leftarrow P(n-2) \Leftarrow \cdots \Leftarrow P(2) \Leftarrow P(1)$.

$q \in \mathbb{Q}^+ : \; P(q) \Leftarrow P(q/2) \Leftarrow P(q/4) \Leftarrow \cdots$ Never hit a base case!

8.2 Real numbers

We saw earlier that we can't solve $x^2 = 2$ in $\mathbb{Q}$.

We want to have numbers like $\sqrt{2}$, since $\sqrt{2}$ is the length of the hypotenuse of the triangle



So there are gaps in $\mathbb{Q}$ and we want to fill them.

Definition We're (informally) going to define real numbers ($\mathbb{R}$) in terms of their decimal expansions: $n.a_1a_2a_3\cdots$ where $n \in \mathbb{Z}$ and $a_k \in \{0, 1, \ldots, 9\}$ for all $k \in \mathbb{N}$. (The meaning of $n.a_1a_2a_3\cdots$ is $n + a_1 \times 10^{-1} + a_2 \times 10^{-2} + a_3 \times 10^{-3} + \cdots$)

Look at the decimal expansion of some rational numbers:

$1/4 = 0.25$, $2/3 = 0.6666\cdots$, $5/6 = 0.8333\cdots$, $6/7 = 0.571428571428$

and $5/11 = 0.454545\cdots$

It can be proved that

(*) x is rational $\Leftrightarrow$ decimal expansion of x is eventually periodic.

(In the case of $1/4$ I can pad out the expansion with 0s, thus: $0.25000\ldots$)

Quiz. Is the decimal expansion of π, $3,1,4,1,5,9,2,6,5$, eventually constant? No. If it were then it would be eventually periodic and hence rational.

One direction of (*) is not too difficult ($\Leftarrow$). "Proof" by example: $0.\underbrace{31}\underbrace{31}\underbrace{31}\ldots$ Note that $0.313131\ldots = 31(10^{-2}+10^{-4}+10^{-6}+\cdots) =$

$$\frac{31}{100}\left(1+10^{-2}+10^{-4}+\cdots\right) = \frac{31}{100}\,\frac{1}{1-1/100} = \frac{31}{100}\times\frac{100}{99} = \frac{31}{99}.$$

The same argument works for any eventually periodic decimal expansion.

Observation: The irrational numbers $\mathbb{R}\setminus\mathbb{Q}$ are characterised by decimal expansions that are <u>not</u> eventually periodic.

We would now need to define the usual arithmetic operations $+,-,\times,\div$ on $\mathbb{R}$. Also need to define $=$ and $\leq$. Equality of real numbers is the equality of their expansions, except for the fact that

$0.999999\ldots = 1.00000$ (and similarly for other numbers ending with $\ldots 99999\ldots$)

So, is there a real number x whose square is 2? Let's try to construct the decimal expansion:

- $1^2 < 2 < 2^2$, so $x = 1.\ldots$
- $1.4^2 < 2 < 1.5^2$, so $x = 1.4\ldots$
- $1.41^2 < 2 < 1.42^2$, so $x = 1.41\ldots$
- $1.414^2 < 2 < 1.415^2$, so $x = 1.414\ldots$

In principle, we can repeat this to find as many digits as required. After doing this we would need to show that $x^2 = 2$. We can also show that $\sqrt{2}$ is the limit of the sequence $1, 1.4, 1.41, 1.414, \ldots$

8.3 Upper bounds

A maximum of a set X of numbers is an element $y \in X$ such that $x \leq y$ for all $x \in X$.

All finite sets have a maximum, and some infinite sets as well, such as $[0,1]$ (its maximum element is 1). Write this as $\max(\ )$:

$$\max\{-3, 2, 7, 9\} = 9 \qquad \max[0,1] = 1.$$

The set of all natural numbers $\mathbb{N}$ does not have a maximum, and neither does $\{x \in \mathbb{Q} : 1 < x < 2\}$, but for different reasons.

__Definition__ Suppose $X \subseteq \mathbb{R}$ and $u \in \mathbb{R}$. We say that u is an __upper bound__ for X if $x \leq u$ for all $x \in X$. If such a u exists we say that X is __bounded above__.

So, $\mathbb{N}$ is not bounded above and so does not have a maximum; $\{x \in \mathbb{Q} : 1 < x < 2\}$ is bounded above but still does not have a maximum.

__Examples__

- Any finite subset of $\mathbb{Z}$ is bounded above and has a maximum.
- $\{-1, -2, -3, -4, \dots\}$ is bounded above (by 0) and has a maximum (-1).
- The set of prime numbers is not bounded above and hence does not have a maximum.