Recap

- Images and inverse images. $f: A \to B$, $C \subseteq A$ and $D \subseteq B$. Then $f(C) = \{f(c) : c \in C\}$ and $f^{-1}(D) = \{a \in A : f(a) \in D\}$. f^{-1} is always defined, $f^{-1}(f(C)) \neq C$ and $f(f^{-1}(D)) \neq D$ in general!
- Relations; reflexive, symmetric, antisymmetric, transitive.

Quiz Let X be any set. Find a relation on X that is reflexive, symmetric and antisymmetric.

Types of relation.

Partial order: reflexive, antisymmetric, transitive. e.g. $\subseteq$, $\mid$ (divides)

Equivalence relation: reflexive, symmetric, transitive;

e.g., $\mathbb{N}$ with the relation $\equiv$ defined by $n \equiv m$ iff $3 \mid (n-m)$.

Transitivity: $n \equiv m$ and $m \equiv k$ then $3 \mid n-m$ and $3 \mid m-k$. Then $3 \mid (n-m)+(m-k)$, i.e. $3 \mid n-k$, and $n \equiv k$. Note that $\equiv$ partitions $\mathbb{N}$ into $\underbrace{\{1, 4, 7, 10, 13, \dots\}}_{\equiv}$, $\underbrace{\{2, 5, 8, 11, 14, \dots\}}_{\equiv}$ and $\underbrace{\{3, 6, 9, 12, 15, \dots\}}_{\equiv}$

7.2 Sequences

Definition: ordered list of elements of some set X, say $a_1, a_2, a_3, a_4, \ldots$ where $a_i \in X$. We can write the sequence as $(a_k)_{k=1}^{\infty}$ or $(a_k)_{k \in \mathbb{N}}$. In this module all sequences are infinite.

Ways to specify a sequence:

- If there is a strong pattern, we can give the first few terms.
- We can give a formula or rule for a_k.
- We can sometimes describe the sequence in words.

Examples

- $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ or $(2^{-k})_{k=1}^{\infty}$
- $1, -1, 1, -1, 1, -1, \ldots$ or $((-1)^{k-1})_{k=1}^{\infty}$
- $3, 1, 4, 1, 5, 9, 2, 6, 5, \ldots$ digits of the decimal expansion of π.
- $1, 4, 9, 16, 25, 36, \ldots$ or $(k^2)_{k=1}^{\infty}$
- $\emptyset, \{2\}, \{3\}, \{2\}, \{5\}, \{2,3\}, \{7\}, \{2\}, \{3\}, \{2,5\}, \{11\}, \{2,3\}, \ldots$
  $= (a_k)_{k=1}^{\infty}$ where $a_k = \{\text{prime factors of } k\}$.

<u>Definition</u> A subsequence of $(a_k)_{k=1}^{\infty}$ is obtained by deleting some of the terms of $(a_k)_{k=1}^{\infty}$, while retaining the original order. The result should be infinite

- $-1, -1, -1, -1, \ldots$ is a subsequence of $\left((-1)^{k-1}\right)_{k=1}^{\infty}$.
- $1, 2, 4, 7, 8, 11, 13, 14, 16, \ldots$ subsequence of $\mathbb{N}$ after deleting numbers divisible by 3 or 5.
- $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \ldots$ subsequence of $\left(\frac{1}{k}\right)_{k=1}^{\infty} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$
- $1, 1, 1, 1, 1, 1, \ldots$ subsequence of the decimal expansion of π? (maybe!); subsequence of the decimal expansion of $1/7 = 0.142857\overline{142857}\ldots$ (yes!). NB repeats!

<u>Definition</u> A sequence of numbers $(x_k)_{k=1}^{\infty}$ is

- increasing if $x_k < x_{k+1}$ for all $k \in \mathbb{N}$,
- decreasing if $x_k > x_{k+1}$ for all $k \in \mathbb{N}$,
- weakly increasing if $x_k \leq x_{k+1}$ for all $k \in \mathbb{N}$,
- weakly decreasing if $x_k \geq x_{k+1}$ for all $k \in \mathbb{N}$,

- constant if $x_k = x_{k+1}$ for all $k \in \mathbb{N}$.

<u>Examples</u>
- $\left(\frac{1}{k}\right)_{k=1}^{\infty} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4},$ is decreasing.
- $\left((-1)^{k-1}\right)_{k=1}^{\infty}$ is neither increasing nor decreasing, but has a constant subsequence: $1, 1, 1, 1, \ldots$
- $\left((-1)^{k-1}\frac{1}{k}\right)_{k=1}^{\infty} = 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \ldots$ is neither increasing nor decreasing, but it has increasing and decreasing subsequences: $-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{6}, -\frac{1}{8}, \ldots$ and $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$.