Recap

- Bijective $\equiv$ Injective and surjective $\equiv$ invertible.
- Restriction f_D of a function f to a subset D of its domain.
- Composition of functions $f: A \to B$ and $g: B \to C$ is $g \circ f: A \to C$.

Theorem 6-3 Let A, B be finite sets and $f: A \to B$. Then:

(a) If f is injective then $|A| \leq |B|$

(b) If f is surjective then $|A| \geq |B|$

(c) If f is bijective then $|A| = |B|$.

Proof (a) Enumerate the elements of the domain $A = \{a_1, a_2, \cdots, a_n\}$. Since f is injective, $f(a_1), f(a_2), \cdots, f(a_n)$ are all distinct, and are elements of B, so $|B| \geq n = |A|$.

(b) Enumerate the elements of the codomain $B = \{b_1, b_2, \cdots, b_m\}$. Since f is surjective, there exist $a_1, \ldots, a_m \in A$ s.t. $f(a_i) = b_i$ for all $i = 1, 2, \cdots, m$. These elements must be distinct, so $|A| \geq m = |B|$.

(c) Immediate from (a) and (b). $\boxtimes$

Note. Earlier in the module we saw a bijective function $f: \mathbb{Z} \to \mathbb{N}$. This suggests that, however cardinality is defined for infinite sets, we should have $\mathbb{Z}$ and $\mathbb{N}$ with the same cardinality!

6.6 Images and inverse images

__Definition__ Suppose $f: A \to B$ is a function, and $C \subseteq A$. The image of C under f is $f(C) = \{f(c) : c \in C\}$.

__Examples__ • $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Then $f([1,2]) = [1,4]$, and $f([-1,1]) = [0,1]$

• $f: \mathcal{P}(\{a,b,c\}) \to \mathbb{N} \cup \{0\}$, $f(X) = |X|$. Then $f(\{\{a,b\},\{b\},\{c\}\}) = \{1,2\}$.

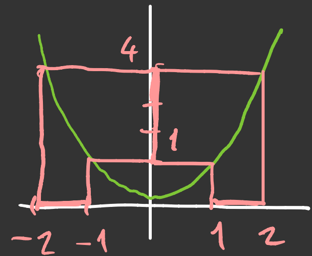
• In general, for $f: A \to B$, $f(\emptyset) = \emptyset$, $f(A) = \text{range}(A)$. If f is injective, then $|f(A)| = |A|$. If f is surjective, then $f(A) = B$.

__Definition__ Suppose $f: A \to B$ and $D \subseteq B$. The inverse image of D under f is $f^{-1}(D) = \{a \in A : f(x) \in D\}$.

NB. This makes sense even if f does not have an inverse!

__Examples__ • $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Then $f^{-1}([1,4]) = [-2,-1] \cup$

$[1,2]$.



- $f: \mathcal{P}(\{a,b,c\}) \to \mathbb{N} \cup \{0\}$, $f(X) = |X|$. Then $f^{-1}(\{0,2,4\}) = \{\emptyset, \{a,b\}, \{a,c\}, \{b,c\}\}$.

In general for $f: A \to B$: $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(B) = A$.

The inverse image behaves well with respect to the set operations $\cap$ and $\cup$, etc. Suppose $f: A \to B$ and D and D' are subsets of B. Then $f^{-1}(D \cap D') = f^{-1}(D) \cap f^{-1}(D')$, $f^{-1}(D \cup D') = f^{-1}(D) \cup f^{-1}(D')$ and $f^{-1}(D \triangle D') = f^{-1}(D) \triangle f^{-1}(D')$.

Quiz. Only one of the above is true with f replacing f^{-1}. E.g., Is it the case that $f(C \cap C') = f(C) \cap f(C')$ where $C, C' \subseteq A$?

NB f and f^{-1} (inverse image and image) are not inverses to each other.

- $f: \mathbb{Z} \to \mathbb{Z}$, $f(n) = n^2$. Then $f^{-1}(f(\{1,2\})) = f^{-1}(\{1,4\}) = \{-2,-1,1,2\} \neq \{1,2\}$. Also $f(f^{-1}(\{1,3,4\})) =$

$f(\{-2,-1,1,2\}) = \{1,4\} \neq \{1,3,4\}$

Quiz: What property of f ensures $f^{-1}(f(C)) = C$? $f(f^{-1}(D)) = D$?

- What happens if we iterate $f, f^{-1}, f, f^{-1}, \ldots$? Quite hard!

7.1 Relations

Definition Let X be a set. A relation R on X is a property that may or may not hold for pairs (x,y) of elements of X; $x R y$ ("x is related to y") may be true or false.

Examples
- $\leq$ is a relation on $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$: $1 \leq 2$, $e \leq \pi$, $2 \not\leq 1$
- $|$ (divides) is a relation on $\mathbb{N}$. $2 | 4$, $2 | 2$, $4 \nmid 2$.
- If S is a set then $\subseteq$ is relation on $\mathcal{P}(S)$. If $S = \{a,b,c\}$ then $\emptyset \subseteq \{a,b\}$, $\{c\} \subseteq \{a,c\}$, $\{a,b\} \not\subseteq \{b,c\}$.
- If S is a finite set then the relation $\equiv$ is defined on $\mathcal{P}(S)$ by $X \equiv Y$ iff $|X| = |Y|$.

Note that we denote the negation of R by $\not R$

Definition A relation R on a set X is

- reflexive: if xRx for all $x \in X$.
- symmetric: if $xRy \Rightarrow yRx$ for all $x, y \in X$.
- antisymmetric: if xRy and $yRx \Rightarrow x = y$, for all $x, y \in X$
- transitive: if xRy and $yRz \Rightarrow xRz$, for all $x, y, z \in X$

Examples

- $X = \mathbb{R}$, with relation $\leq$.
  - reflexive ✓ $x \leq x$ for all $x \in \mathbb{R}$
  - symmetric ✗ $1 \leq 2$ but not $2 \leq 1$.
  - antisymmetric ✓ $x \leq y$ and $y \leq x \Rightarrow x = y$
  - transitive ✓ $x \leq y$ and $y \leq z \Rightarrow x \leq z$.

  partial order (p.o.)

- $X = \mathbb{N}$, with relation $|$.
  - reflexive ✓ $n \mid n$ for all $n \in \mathbb{N}$
  - symmetric ✗ $2 \mid 4$ but $4 \nmid 2$.
  - antisymmetric ✓ $n \mid m$ and $m \mid n \Rightarrow n = m$
  - transitive ✓ $n \mid m$ and $m \mid k \Rightarrow n \mid k$.

  p.o.

- $X = \mathcal{P}(\mathbb{N})$ with relation $\subseteq$
  - reflexive ✓ $A \subseteq A$ for all $A \subseteq \mathbb{N}$
  - symmetric ✗ $\{1\} \subseteq \{1,2\}$ but $\{1,2\} \not\subseteq \{1\}$.
  - antisymmetric ✓ $A \subseteq B$ and $B \subseteq A \Rightarrow A = B$
  - transitive ✓ $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$.

  p.o.

- $X = \mathcal{P}(\{a,b,c\})$ with relation $\equiv$ defined by $A \equiv B$ iff $|A| = |B|$.
  - reflexive ✓ $|A| = |A| \Rightarrow A \equiv A$
  - symmetric ✓ $A \equiv B \Rightarrow |A| = |B| \Rightarrow |B| = |A| \Rightarrow B \equiv A$.
  - antisymmetric ✗ $\{a,b\} \equiv \{b,c\}$, $\{b,c\} \equiv \{a,b\}$ but $\{a,b\} \neq \{b,c\}$.
  - transitive ✓ $A \equiv B$ and $B \equiv C \Rightarrow |A| = |B| = |C| \Rightarrow A \equiv C$

  (equivalence relation)

- $X = \mathbb{Z}$ $\quad nRm$ if $|n-m| = 1$.
  - reflexive ✗ $|1-1| = 0 \neq 1$, so $1 \not R 1$.
  - symmetric ✓ $nRm \Rightarrow |n-m| = 1 \Rightarrow |m-n| = 1 \Rightarrow mRn$
  - antisymmetric ✗ $1R2$ and $2R1$ but $1 \neq 2$.

- transitive $^{\times}$ $\quad 1 \, R \, 2$ and $2 \, R \, 3$ but $1 \not R \, 3$.

  (nothing special!)