<u>Recap</u> • Function $g: B \to A$ is an inverse to function $f: A \to B$ if $g(f(a)) = a$ for all $a \in A$ and $f(g(b)) = b$ for all $b \in B$.

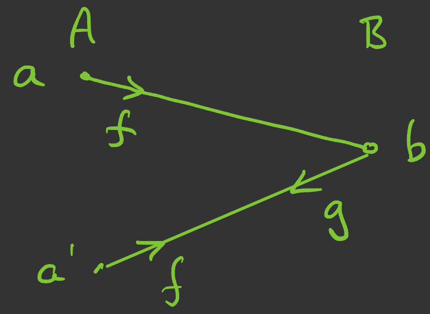
• If f has an inverse then that inverse is unique; we denote the inverse by f^{-1}.

Owing to a fire alarm we only managed to prove the $(\Rightarrow)$ part of the following theorem (up to $\diamond$). We pick up the threads now.

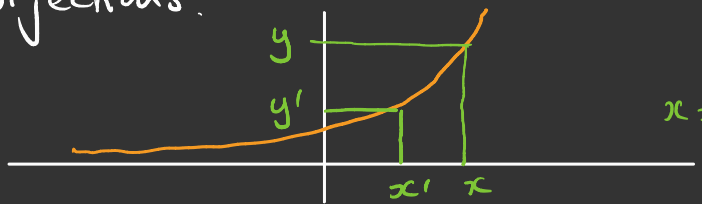
<u>Theorem 6.2</u> Suppose $f: A \to B$ is a function. Then f is invertible iff f is bijective.

<u>Proof</u> $(\Rightarrow)$ Suppose f is invertible with inverse $g: B \to A$. If $a_1, a_2 \in A$ are such that $f(a_1) = f(a_2)$, then applying g to both sides, $a_1 = g(f(a_1)) = g(f(a_2)) = a_2$; so f is injective. Now suppose $b \in B$ is arbitrary. Then $a = g(b)$ has the property $f(a) = f(g(b)) = b$, so f is surjective. $\diamond$

$\Leftarrow$ Suppose $f: A \to B$ is bijective. We define a function $g: B \to A$ as follows. Let $b \in B$. Since f is surjective, there exists $a \in A$ such that $f(a) = b$. Then let $g(b) = a$. Note that $f(g(b)) = f(a) = b$. This is one half of the definition of invertible. Now following the diagram, $f(a) = f(a')$ $= f(g(f(a)))$. (This is because $f(g(b)) = b$.) Since f is injective, we must have $g(f(a)) = a$. This is the other half of the definition of invertible. $\boxtimes$



<u>Example</u> $\exp: \mathbb{R} \to (0,\infty)$, $\ln: (0,\infty) \to \mathbb{R}$ are inverses, i.e., $\exp(\ln(y)) = y$ and $\ln(\exp(x)) = x$. We deduce that exp and ln are bijections.

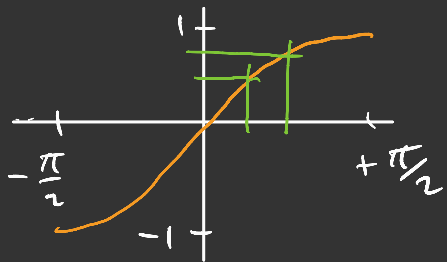


$y = \exp(x)$

$x \neq x' \Rightarrow y \neq y'$.

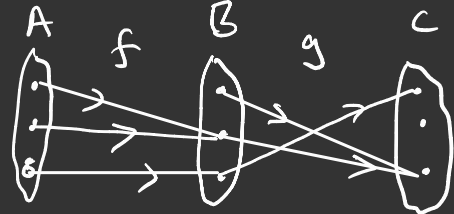
Definition Suppose $f: A \to B$ and $D \subseteq A$. The restriction of f to D is the function $f|_D : D \to B$ defined by $f|_D(a) = f(a)$ for all $a \in D$.

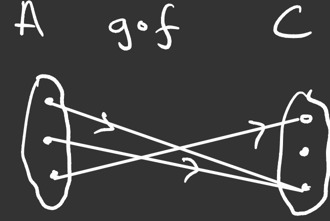
Example $\sin : \mathbb{R} \to [-1, 1]$ is not injective; for example, $\sin(0) = 0 = \sin(\pi)$. However, letting $D = [-\pi/2, \frac{\pi}{2}]$, $\sin|_D$ is injective. Also, $\sin|_D$ is surjective, so by Theorem 6.2 $\sin|_D$ has an inverse, namely arcsin.



Definition Let A, B, C be sets, $f: A \to B$, $g: B \to C$. Then the composition of f and g is the function $g \circ f : A \to C$ defined by $(g \circ f)(a) = g(f(a))$ for all $a \in A$.

Example (picture)

• 



• Alternative way of saying g is the inverse of $f : A \to B$ is $g \circ f = id_A$ (id_A is the identity function on A, i.e. $id_A(x) = x$ for $x \in A$) and $f \circ g = id_B$.

• If $f, g : \mathbb{R} \to \mathbb{R}$ are defined by $f(x) = x^2$ and $g(x) = -x$. Then $g \circ f(x) = g(x^2) = -x^2$ but $f \circ g(x) = f(-x) = +x^2$. Order of composition matters!

6.5 Bijections and cardinality

<u>Theorem 6.3</u> Let A, B be finite sets and $f : A \to B$. Then:

(a) If f is injective then $|A| \leq |B|$

(b) If f is surjective then $|A| \geq |B|$

(c) If f is bijective then $|A| = |B|$.