<u>Recap</u> • Defining functions. Examples and non-examples of functions.

- Picturing functions.
- The range of a function $f: A \to B$ is $\text{range}(f) = \{f(a) : a \in A\}$.
- Injective ($a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$), surjective ($\text{range}(f) = \text{codomain}(f)$), bijective (injective and surjective).

<u>Examples</u> • $f: \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^2$. Not injective ($f(2) = 4 = f(-2)$). Not surjective ($f(?) = -1$).

- $f: \mathbb{N} \to \mathbb{N}$, $f(n) = n^2$. Not surjective ($f(?) = 3$). Injective: Assume $f(n) = f(m)$. Then $n^2 = m^2$, i.e., $n^2 - m^2 = 0$ and $(n-m)(n+m) = 0$. Since $n+m \neq 0$ it must be the case that $n = m$.
- $f: \mathbb{N} \setminus \{1\} \to \mathbb{N}$. $f(n) = |\{p : p \text{ is prime and } p \mid n\}|$. Not injective ($f(6) = |\{2,3\}| = 2 = |\{3,5\}| = f(15)$.) Surjective (Suppose $m \in \mathbb{N}$. Let $n = p_1 p_2 \cdots p_m$ where $p_1 \cdots p_m$ are the first m prime numbers. Then $f(n) = m$.)

- $\sin : \mathbb{R} \to [-1,1]$. Not injective ($\sin(0) = 0 = \sin(\pi)$). It is surjective (IVT = Intermediate Value Theorem).
- $f : \mathcal{P}(X) \to \mathcal{P}(X)$. $f(A) = X \setminus A$, for all $A \subseteq X$. Injective (Suppose $f(A) = f(B)$. Then $X \setminus A = X \setminus B \Rightarrow$ $X \setminus (X \setminus A) = X \setminus (X \setminus B) \Rightarrow A = B$. Also surjective (Let B be a subset of X; then $f(X \setminus B) = X \setminus (X \setminus B) = B$.)
- $f : \mathcal{P}(X) \to \{0, 1, 2, \dots, |X|\}$. For $A \subseteq X$, $f(A) = |A|$. Not injective (check), surjective (check).

6.3 Inverses

Definition. Suppose $f : A \to B$. Then $g : B \to A$ is an inverse to f if $g(f(a)) = a$ for all $a \in A$, and $f(g(b)) = b$ for all $b \in B$. If such a function exists, we say that f is invertible.

Examples.
- $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 3$. The inverse $g : \mathbb{R} \to \mathbb{R}$ is $g(y) = \frac{y-3}{2}$. Check this:

$$g(f(x)) = \frac{(2x+3)-3}{2} = \frac{2x}{2} = x \quad \checkmark$$

$$f(g(y)) = 2\left(\frac{y-3}{2}\right) + 3 = y - 3 + 3 = y \checkmark$$

- $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 2n+3$. No inverse! There is no way to define $g(2)$, since $f(g(2))$ is odd.
- $\sin : \mathbb{R} \to [-1, 1]$. No inverse! $g(\sin(0)) = g(0) = 0$ but $g(\sin(\pi)) = g(0) = \pi$; $g(0)$ multiply defined.

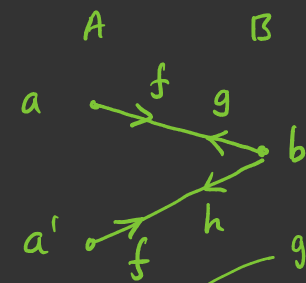
<u>Lemma 6.1</u> Suppose $f : A \to B$ is a function. If f is invertible then the inverse is unique.

<u>Proof</u> Suppose $g, h : B \to A$ are both inverses of f. We aim to show that $g = h$.

Let $b \in B$. Consider $h(f(g(b)))$. Then

$$g(b) = h(f(g(b))) = h(b)$$

$\uparrow$ $h(f(a)) = a$ $\qquad\qquad$ $\uparrow$ $f(g(b)) = b$



$g(b)$

$$g(b) = a = h(f(a)) = a' = h(b)$$

Since $b \in B$ was chosen arbitrarily, g and h agree at all points of their domain. $\boxtimes$

Write f^{-1} for the inverse of f.

Examples • $f : [0,\infty) \to [0,\infty)$, $f(x) = x^2$. Then $f^{-1}(y) = \sqrt{y}$.

• $f : \mathbb{Z} \to \mathbb{N}$ defined by

$$f(n) = \begin{cases} 2n, & \text{if } n > 0; \\ 1-2n, & \text{if } n \leq 0. \end{cases}$$

We have $f^{-1} = g$ where

$$g(m) = \begin{cases} \frac{m}{2}, & \text{if } m \text{ is even}; \\ \frac{1-m}{2}, & \text{if } m \text{ is odd}. \end{cases}$$

Check.
$$g(f(n)) = \begin{cases} g(2n) = \frac{2n}{2} = n, & \text{if } n > 0; \\ g(1-2n) = \frac{1-(1-2n)}{2} = \frac{2n}{2} = n, & \text{if } n \leq 0. \end{cases}$$

Homework: check $f(g(m)) = m$.

Theorem 6.2 Suppose $f : A \to B$ is a function. Then f is invertible iff f is bijective.

Proof ($\Longrightarrow$) Suppose f is invertible with inverse $g : B \to A$. If $a_1, a_2 \in A$ are such that $f(a_1) = f(a_2)$, then applying g to both sides, $a_1 = g(f(a_1)) = g(f(a_2)) = a_2$; so f is injective. Now suppose $b \in B$ is arbitrary. Then $a = g(b)$ has the property $f(a) = f(g(b)) = b$; so f is surjective.

[To be continued on Friday.]