<u>Recap</u> The general topic was sets.

We completed a brief survey of identities involving set operations.

We introduced an additional operation on sets, Cartesian product:

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

Generally, we consider $((a,b), c)$ and $(a,(b,c))$ to be equivalent, so we write (a,b,c) instead.

We defined cardinality of a (finite) set — just the number of elements it contains. Write the cardinality of A as $|A|$.

<u>5·6 Counting subsets.</u>

• If A, B are sets then $|A \times B| = |A| \times |B|$.

Why is this? A typical element of $A \times B$ is (a,b) where $a \in A$ and $b \in B$. We can select a in A, and then $b \in B$. These two choices are <u>independent</u>; there are $|A|$ choices for a and $|B|$ for b. So the cardinality of $A \times B$ is just $|A| \times |B|$.

• Counting permutations of n objects, e.g. $\{1, 2, \ldots, n\}$. A permutation is a listing $a_1, a_2, \ldots, a_n$ such that each object occurs exactly once. There are n choices for a_1, and then $n-1$ choices for a_2, $n-2$ for a_3, $\dots$, 2 for a_{n-1} and 1 for a_n. The choices are <u>not</u> independent, but the number of choices is unchanging. So the total number of choices is $n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 = n!$ Example: permutations of $\{1,2,3\}$. A typical permutation is $2, 3, 1$.

(2: 3 choices; 3: 2 choices; 1: 1 choice) $\Longrightarrow$ $3 \times 2 \times 1 = 3!$ choices in total.

<u>Definition</u> Power set. If X is a set then the power set $\mathcal{P}(X)$ of X is the set of all subsets of X: $\mathcal{P}(X) = \{A : A \subseteq X\}$.

Ex. $X = \{a, b, c\}$. Then

$$\mathcal{P}(X) = \{\underbrace{\emptyset}_{1}, \underbrace{\{a\}, \{b\}, \{c\}}_{3}, \underbrace{\{a,b\}, \{a,c\}, \{b,c\}}_{3}, \underbrace{\{a,b,c\}}_{1}\}$$

Note symmetry!

Note that $|\mathcal{P}(X)| = 8 = 2^3$

$|\mathcal{P}(\{a,b\})| = 4 = 2^2$

$$|\mathcal{P}(\{a\})| = 2 = 2^1$$
$$|\mathcal{P}(\emptyset)| = 1 = 2^0$$

Theorem 5.2 Suppose X is a finite set with $|X| = n$. Then $|\mathcal{P}(X)| = 2^n$.

Proof Label the elements of X: $x_1, x_2, \ldots, x_n$. Let us select a subset of X as follows. Call the subset $A \subseteq X$. Decide whether x_1 is included in the subset; then decide whether x_2 is in A or not. Then $x_3 \in A$ or $x_3 \notin A$, and so on up to x_n. At each step there are two choices, and these are all independent. So the total number of choices is $\underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ times}} = 2^n$. ⊠

5.7 Counting subsets of a particular size

Definition. Suppose X is a set and $k \in \mathbb{Z}$. A k-element subset of X is a subset $A \subseteq X$ with $|A| = k$. If $k \geq 0$ and $|X| = n$, then $\binom{n}{k}$ is the number of k-subsets of X.

We have already seen that $\binom{3}{0}=1$, $\binom{3}{1}=3$, $\binom{3}{2}=3$, $\binom{3}{3}=1$.
We read $\binom{n}{k}$ as "n choose k". In general, we call these symbols "binomial coefficients".

Examples: $\binom{n}{-1}=0$, $\binom{n}{0}=1$, $\binom{n}{1}=n$, ..., $\binom{n}{n-1}=n$, $\binom{n}{n}=1$.

Theorem 5.3 Suppose n, k are integers, and that $k \geq 0$ and $k \leq n$. Then
$$\binom{n}{k}=\frac{n!}{k!(n-k)!}.$$

[$n=3$ $\binom{n}{0}=\frac{3!}{0!(3-0)!}=1$, $\binom{n}{1}=\frac{3!}{1!(3-1)!}=3$, ...]

Proof: Count the ways to select a k-subset from an n-element set X. Let's say this k-subset is $\{a_1, a_2, \ldots, a_k\}$. An easier problem is to select an ordered tuple $(a_1, a_2, \ldots, a_k)$. The number of ways to select a_1 is n. There are $n-1$ ways to select a_2, $n-2$ ways to select a_3, ..., $n-k+1$ ways to select a_k.
$n-k+1 = n-(k-1)$

The total number of ways to select $(a_1, a_2, \ldots, a_k)$ is

$$n \times (n-1) \times \cdots \times (n-k+1) = \frac{n \times (n-1) \times \cdots \times 2 \times 1}{(n-k) \times (n-k-1) \times \cdots \times 2 \times 1} = \frac{n!}{(n-k)!}$$

However, this expression overcounts the number of (unordered) k-subsets. Each subset $\{a_1, \ldots, a_k\}$ will occur $k!$ times as an ordered tuple. So the overcounting is by a factor $k!$

So the number of k-subsets is $\frac{n!}{k!(n-k)!}$. ☒

How does this look for $n=5$ and $k=3$? We have $\frac{5!}{2!} = 60$ ordered triples from a set of 5 elements, say $\{1, 2, \ldots, 5\}$:

$$\underbrace{(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)}_{\text{these are equivalent}}, \ldots, (5,4,3).$$

The number of <u>unordered</u> 3-subsets is $\frac{5!}{3!\,2!}$.

If we compute a binomial coefficient, we can make use of cancellation.

E.g. $\binom{9}{5} = \frac{9!}{5!\,4!} = \frac{9\times8\times7\times6\times5\times4\times3\times2\times1}{5\times4\times3\times2\times1\times4\times3\times2\times1} = \frac{9\times2\times7}{1}$

$= 126$

Quiz. How many ways are there to partition a set X, with $|X| = n$, into three sets A, B, C such that $|A| = k$ and $|B| = \ell$. ($k \geq 0$, $\ell \geq 0$, $k+\ell \leq n$)?

There are $\binom{n}{k}$ ways to select A. Then are $\binom{n-k}{\ell}$ ways to select B. There is 1 way to select C. Overall, there are $\binom{n}{k}\binom{n-k}{\ell}$ choices for the partition of X into A, B, C.

↑ choices for A ↑ choices for B

Note that $\binom{n}{k}\binom{n-k}{\ell} = \frac{n!}{k!\,(n-k)!} \times \frac{(n-k)!}{\ell!\,(n-k-\ell)!}$

$$= \frac{n!}{k!\,\ell!\,(n-k-\ell)!}$$

6. Functions

6.1 Definition Let A, B, be sets. A function is a rule that associates an element $b \in B$ for every $a \in A$. We write $f: A \to B$ and say that A is the domain and B the codomain. The element of B that is associated to $a \in A$ is called $f(a)$. If f is understood we may write $a \mapsto b$, and say that a maps to b.