Recap Defining sets:

- Listing the elements. $\{1, 2, 3, \dots n\}$
- Describing the set in words. The set of employees of QMUL
- Standard sets: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \emptyset$
- Specifying a property that the elements have. $\{n \in \mathbb{N} : P(n)\}$
- Ditto, but with an operation applied to objects satisfying the property. $\{n^2 : n \in \mathbb{N}, P(n)\}$

Subsets, disjoint sets.

Operations on sets: $\cup, \cap, \setminus, \Delta$.

Various identities involving these operations: commutativity, associativity, distributivity.

We left one identity to check: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

($\subseteq$) Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. There are two cases: $x \in B$ or $x \in C$. In the first case, $x \in B$ and hence $x \in A \cap B$. Therefore $x \in (A \cap B) \cup (A \cap C)$. Second case similar.

($\supseteq$) Let $x \in (A \cap B) \cup (A \cup C)$. There are two cases: $x \in A \cap B$ or $x \in A \cap C$. In the first case $x \in A$ and $x \in B$, and hence $x \in B \cup C$. Thus $x \in A \cap (B \cup C)$. In the second, $x \in A$ and $x \in C$ and hence $x \in B \cup C$. Therefore $x \in A \cap (B \cup C)$.

From ($\subseteq$) and ($\supseteq$) we deduce equality. $\boxtimes$

By associativity, we don't need brackets in $A \cap B \cap C$, however, we need brackets for $A \cap (B \cup C) \neq (A \cap B) \cup C$.

5.4 Cartesian product. We write (a, b) for the ordered pair a followed by b. Order is important, and $a = b$ is allowed.

<u>Definition</u> If A and B are sets the Cartesian product, $A \times B$, is the set of pairs such that the first item is from A and the second from B. In symbols:

$$A \times B = \{(a,b) : a \in A \text{ and } b \in B\}.$$

Examples: Let $A = \{a,b,c\}$, $B = \{1,2\}$. Then

$$A \times B = \{(a,1),(a,2),(b,1),(b,2),(c,1),(c,2)\} \neq$$
$$B \times A = \{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c)\}.$$

$$A \times \emptyset = \emptyset = \emptyset \times A.$$

Formally, $((a,b),c) = (a,(b,c))$. However we generally regard these as the same and write (a,b,c). If we take a product of a set A with itself we write $A^2 = A \times A$, $A^3 = A \times A \times A$. Example: $\mathbb{R}^3$, Euclidean space.

Quiz. Is it the case that $A \times (B \cup C) = (A \times B) \cup (A \times C)$?

5.5 Cardinality

Definition The cardinality of a finite set A is just the number of elements it contains. We write $|A|$ for cardinality of A.

Examples $|\{1,2,7\}| = 3$

$$|\{2,4,6,\dots,2n\}| = n$$

$$|\emptyset| = 0.$$

$$A \times (B \cup C) \overset{?}{=} (A \times B) \cup (A \times C)$$

Break $=$ into $\subseteq$ and $\supseteq$.

$(\subseteq)$ Let $x \in A \times (B \cup C)$. Then $x = (a,b)$ where $a \in A$ and $b \in B \cup C$. There are two cases (i) $b \in B$ or (ii) $b \in C$.
In case (i) $(a,b) \in A \times B$ and so $(a,b) \in (A \times B) \cup (A \times C)$.
In case (ii) $(a,b) \in A \times C$ and so $(a,b) \in (A \times B) \cup (A \times C)$.
$(\supseteq)$ Exercise.