Recap • Definition of lowest common multiple $\mathrm{lcm}(a,b)$ of numbers a, b.

- Suppose $a, b, n \in \mathbb{N}$. If $a \mid n$ and $b \mid n$ then $\mathrm{lcm}(a,b) \mid n$.
- $ab = \mathrm{lcm}(a,b)\,\gcd(a,b)$. This gives a way to compute $\mathrm{lcm}(a,b)$.

This gives us a way of computing $\mathrm{lcm}(a,b)$. Example:

$\gcd(28,42) = \gcd(42,28) = \gcd(28,14)$ $[= \gcd(14,0)]$ $= 14$. So, by Theorem 4.8:

$$\mathrm{lcm}(28,42) = \frac{28 \times 42}{\gcd(28,42)} = \frac{28 \times 42}{14} = 84.$$

5.1 Sets: definitions, notation, examples

A set is a collection of objects. The objects are called the elements of the set. If we have a set S and x is an element of S then we write $x \in S$. If x is

$\in$ membership, ε epsilon

not an element of S we write $x \notin S$. Sets are considered equal if they contain the same elements.
A finite set can be written by listing its elements:

$$\{MTH4000, MTH4213, MTH4300, MTH4500\}$$

set brackets

commas.

Ordering is not significant:

$$\{1,2,3,4\} = \{2,4,1,3\}$$

Repeats are not significant

$$\{1,2,3,4\} = \{1,2,2,2,3,4\}.$$

Ways to define sets:

- Listing the elements can work for infinite sets if there is a clear pattern to the elements:

  $\{2, 4, 6, 8, \ldots\}$ = set of even numbers.

  $\{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$

  $= \{0, 2, -2, 4, -4, 6, -6, \ldots\}$.

  Can also be used for sets of indeterminate size:

  $\{1, 2, 3, \ldots, n\}$ = first n natural numbers.

- Describe the set in words:

  the set of all students taking MTH4213.

- Some important sets have special names:

  $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\emptyset$ ← empty set, $\emptyset$ phi.

- Giving a rule for membership:

set bracket $\rightarrow \{ n : \underline{n \text{ is even}} \} \leftarrow$ set bracket

colon, $|$ $\uparrow$ statement involving n (some $P(n)$)

Other examples:

$\{ x : x \in \mathbb{R} \text{ and } x > 0 \}$ = positive real numbers

Alternatively,

$\{ x \in \mathbb{R} : x > 0 \}$ = ditto.

The condition $P(n)$ may be more complex:

$\{ x \in \mathbb{N} : \text{there exists } k \in \mathbb{N} \text{ such that } n = 2k \}$

= set of even numbers.

- Like the previous, except we apply an operation to the left hand side of the colon:

$\{ n^2 : n \in \mathbb{N} \}$ = set of square numbers

$= \{ m \in \mathbb{N} : \text{there exists } n \in \mathbb{N} \text{ s.t. } m = n^2 \}$.

Rational numbers: $\mathbb{Q} = \{ a/b : a, b \in \mathbb{Z} \text{ and } b \neq 0 \}$

Repetitions are convenient here!

$\{n^2+m^2 : n,m \in \mathbb{N}\}$ = set of numbers expressible as the sum of two squares.

There are repetitions : $5^2+5^2 = 50 = 7^2+1^2$.

5.2 Subsets

Definition A is a subset of B, written $A \subseteq B$, if every element of A is contained in B:

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \cdots$$

"A is not a subset of B" is written $A \nsubseteq B$.

In symbols, $A \subseteq B$ can be written $\forall x .\ x \in A \Rightarrow x \in B$.

$x \notin A \vee x \in B$

$\equiv$

and $A \nsubseteq B$ can be written $\exists x .\ x \in A \wedge x \notin B$.

Quiz: Define $S = \{X : X \notin X\}$. Is $S \in S$?

Suppose $S \in S$; then from definition of S, $S \notin S$.

Suppose $S \notin S$; then from the definition, $S \in S$

$A \subset B$: "A is a strict subset of B" or $A \subseteq B \wedge A \neq B$.

5.3 Set operations

We've seen operations on numbers: $+, \times, -$

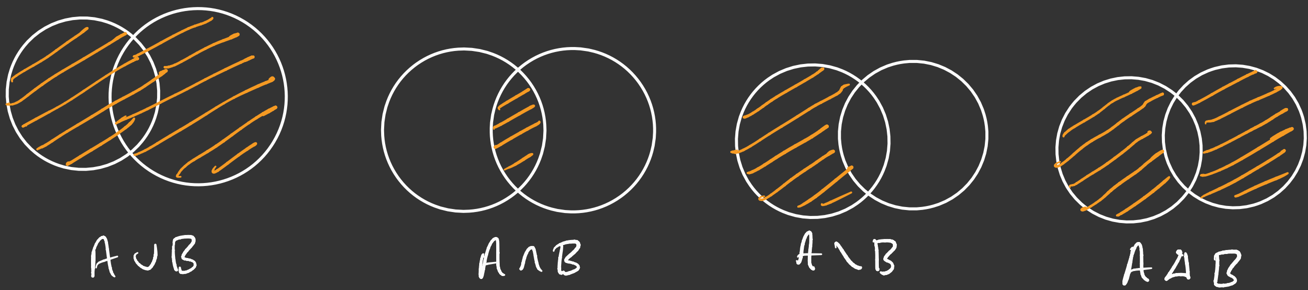
also on statements: and, or, not.

Similarly, we have operations on sets: $\cap, \cup, \Delta, \setminus$.

- Union of A, B, written $A \cup B$, is the set of all objects that are elements of A or B (or both).
- Intersection of A, B, written $A \cap B$, is the set of all objects that are elements of both A and B
- Difference of A, B, written $A \setminus B$, is all things in A but not B.
- Symmetric difference of A, B, written $A \Delta B$ is all things in A or B but not both.

In symbols:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$
$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$
$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$$
$$A \triangle B = \{x : (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)\}$$



$$A \triangle B = A \setminus B \cup B \setminus A = (A \cup B) \setminus (A \cap B)$$

Why not do proofs with Venn diagrams? Only good up to three sets.

Look at the properties of these operations

<u>Proposition 5.1</u> Let A, B, C be sets. Then

(a) $(A \cap B) \cap C = A \cap (B \cap C)$
✓ (b) $(A \cup B) \cup C = A \cup (B \cup C)$
✓ (c) $(A \Delta B) \Delta C = A \Delta (B \Delta C)$

can leave out brackets

(d) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

NB distributivity both ways

<u>Proof</u> (a) left as an exercise.

(b) $(A \cup B) \cup C = \{x : x \in (A \cup B) \text{ or } x \in C\}$

$$= \{x : x \in A \text{ or } x \in B \text{ or } x \in C\}$$

$$= \{x : x \in A \text{ or } x \in (B \cup C)\}$$

$$= A \cup (B \cup C)$$

(c) Proof by truth table.

We have three atomic statements:

$x \in A$, $x \in B$, $x \in C$.

$x \in A$	$x \in B$	$x \in C$	$x \in (A \triangle B) \triangle C$	$x \in A \triangle (B \triangle C)$
F	F	F	F	F
F	F	T	T	T
F	T	F	T	T
F	T	T	F	F
T	F	F	T	T
T	F	T	F	F
T	T	F	F	F
T	T	T	T	T

columns equal!

Since the truth assignments are exhaustive, we see that $x \in (A \triangle B) \triangle C$ is equivalent to $x \in A \triangle (B \triangle C)$. So we have proved the identity.