Special proof technique 3 : Induction

Can be used to prove statements with a positive integer n. Suppose we can prove $P(1)$, and that for $n \geq 2$, $P(n)$ follows from $P(n-1)$. Then we deduce $P(n)$ for all integers n.

- Base case: $n = 1$.
- Inductive step: showing $P(n)$ follows from $P(n-1)$ for all $n \geq 2$.

Theorem 3.9 Suppose n is a positive integer; then

$$\sum_{k=1}^{n} (2k-1) = n^2 .$$

This says

$$\begin{aligned} 1 &= 1^2 & (n=1) \\ 1+3 &= 2^2 & (n=2) \\ 1+3+5 &= 3^2 & (n=3) \end{aligned}$$

Proof by induction shows this infinite set of statements in one go.

Proof Let $P(n)$ by the statement : $\sum_{k=1}^{n}(2k-1) = n^2$. $(*)$

We prove $P(n)$ by induction on n.

Base case $(n=1)$: $P(1)$ true since both sides of $(*)$ are 1.

Assume $n \geq 2$:

$$\sum_{k=1}^{n}(2k-1) = \sum_{k=1}^{n-1}(2k-1) + (2n-1)$$

$$= (n-1)^2 + (2n-1) \quad (\text{using } P(n-1))$$

$$= n^2 - 2n + 1 + 2n - 1$$

$$= n^2.$$

This establishes $P(n)$, and completes the proof. $\boxtimes$

Informally, why does this work?

$P(1)$ is true by direct checking.

$P(1)$ and $P(1) \Rightarrow P(2)$ ... so $P(2)$ is true

$P(2)$ and $P(2) \Rightarrow P(3)$ ... so $P(3)$ is true.

<u>Theorem 3.10</u> Suppose n is a positive integer. Then $8^n - 3^n$ is divisible by 5.

<u>Proof</u> Let $P(n)$ be the statement

"$8^n - 3^n$ is divisible by 5".

We will show $P(n)$ for all n by induction.

Base case $(n=1)$: $P(1)$ holds since $8^1 - 3^1 = 5$ which is divisible by 5.

Now assume $n \geq 2$. Then

$$8^n - 3^n = 8 \times 8^{n-1} - 3 \times 3^{n-1}$$

$$= 3 \times 8^{n-1} - 3 \times 3^{n-1} + 5 \times 8^{n-1}$$

$$= 3(8^{n-1} - 3^{n-1}) + 5 \times 8^{n-1}$$

$$= 3 \times 5k + 5 \times 8^{n-1}$$

using $P(n-1)$: $8^{n-1} - 3^{n-1}$ is divisible by 5

$$= 5(3k + 8^{n-1})$$

which establishes $P(n)$ $\boxtimes$

Tips for inductive proofs:

- Give a name to the statement being proved ($P(n)$ above).
- Recall that $P(n)$ is a statement, not an expression.
- In the induction step, make the inductive hypothesis $P(n-1)$ explicit.
- Each step follows in a clear way.
- Avoid nonsense such as $n = n+1$.

In the break: Let a, b be positive integers with $a > b$.
Is it the case that $a^n - b^n$ is divisible by $a - b$?
Also is there a more enlightening proof?

Note that $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1})$

If a, b are integers, $(a-b)$ divides $(a^n - b^n)$. Equivalently:

$$a^n - b^n = (a-b) \sum_{k=0}^{n-1} a^{n-k-1} b^k.$$

This can be shown using techniques from the summation section of the course. Try it!

In strong induction we prove the base case $(n=1)$.
For the inductive step we show $P(1), P(2), \ldots, P(n-1)$ together imply $P(n)$. (Don't <u>really</u> need the base case!)

Fibonacci sequence: $F_1, F_2, F_3, \ldots$ defined by

$F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

(Sequece is $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$)

<u>Theorem 3.11</u> Suppose n is a positive integer. Then $F_n < 2^n$.

<u>Proof</u> Let $P(n)$ denote the statement $F_n < 2^n$.

False start. Base case $(n=1)$, $F_1 = 1 < 2^1$. ✓

Now assume $n \geq 2$: $F_n = F_{n-1} + F_{n-2}$ ... whoops!

We prove $P(n)$ by strong induction.

Base cases $(n=1)$: $F_1 = 1 < 2^1$ so $P(1)$ holds

$(n=2)$: $F_2 = 1 < 2^2$ so $P(2)$ holds.

Now assume $n \geq 3$. Then

$$F_n = F_{n-1} + F_{n-2} < 2^{n-1} + 2^{n-2} < 2^{n-1} + 2^{n-1} = 2^n.$$

by defn

by $P(n-1)$ and $P(n-2)$

So $P(n)$ holds. $\boxtimes$

Theorem 3.12 Suppose n is a positive integer. Then

$P(n)$: F_n is even $\Longleftrightarrow$ n is divisible by 3.

Proof Prove $P(n)$ by strong induction.

Base cases $(n=1)$: F_1 is odd and n is not divisible by 3
$(n=2)$: F_2 is odd and n " " " " "

Suppose $n \geq 3$. Then by definition, $F_n = F_{n-1} + F_{n-2}$

Case analysis:

$n = 3k$ for some k. F_{n-1} is odd F_{n-2} is odd (*)
and $F_n = F_{n-1} + F_{n-2}$ is even.

$n = 3k+1$: F_{n-1} is even and F_{n-2} is odd (*)
and $F_n = F_{n-1} + F_{n-2}$ is odd

$n = 3k+2$: F_{n-1} is odd and F_{n-2} is even (*)
and $F_n = F_{n-1} + F_{n-2}$ is odd

(*) Using the inductive hypotheses $P(n-1)$, $P(n-2)$.

Putting the cases together: $P(n)$ holds. $\boxtimes$