SPA7010 Example Problems 3
(13 Feb 2020)

Gravitational potentials, density distributions and masses

Under spherical symmetry, the gravitational potential Φ, the mass interior to a particular radius and the density ρ are all functions of the radius r from the centre alone.

Converting between $\Phi(r)$ and the density $\rho(r)$ can be done using Poisson's equation

$$\nabla^2\Phi \;=\; 4\pi G\rho$$

(a general result for any gravitational field and any distribution of mass).

In the case of spherical symmetry, $\nabla^2\Phi$ is a function of r alone (see Lecture 1) with

$$\nabla^2\Phi \;=\; \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)$$

Also, for spherical symmetry, the following relations exist between the gravitational potential $\Phi(r)$, the mass $M(r)$ within a radius r, and density $\rho(r)$:

$$\frac{\mathrm{d}M}{\mathrm{d}r} \;=\; 4\pi r^2\rho \qquad \text{and} \qquad M(r) \;=\; \frac{r^2}{G}\frac{\mathrm{d}\Phi}{\mathrm{d}r}\ .$$

The first of these two equations is just the equation of conservation of mass.

We can therefore convert from $\Phi(r)$ to $M(r)$ and to $\rho(r)$ by differentiation, and from $\rho(r)$ to $M(r)$ and to $\Phi(r)$ by integration.

For example:

1. Given the density $\rho(r)$:

To find the mass, $M(R)$ inside some radius, R, consider a thin spherical shell of radius r and thickness dr. The mass in the shell will be

$$\mathrm{d}M \;=\; 4\pi\, r^{\,2}\, \rho(r)\, \mathrm{d}r$$

So, integrating from $r = 0$ to $r = R$:

$$M(R) = \int_0^{M(R)} \mathrm{d}M' \;=\; \int_0^R 4\pi\, r^{\,2}\, \rho(r)\, \mathrm{d}r$$

Then integrate again to find the potential:

$$\Phi = \int_0^R \frac{GM(r)}{r^2}\,\mathrm{d}r$$

2. Given the potential $\Phi(r)$:

To find the mass, M(r), we can use:

$$M(r) \;=\; \frac{r^2}{G}\frac{\mathrm{d}\Phi}{\mathrm{d}r} \;.$$

Then differentiate again to get the density, $\rho(r)$, from Poisson's equation:

$$4\pi G\rho \;=\; \nabla^2\Phi \;=\; \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)$$

Or use Poisson's equation to go directly from potential to density.

Problem 3.1

The gravitational potential in a spherically-symmetric galaxy is given by

$$\Phi(r) \;=\; 2\pi G\rho_0 a^2\left(\ln(r^2+a^2) \;+\; \frac{2a}{r}\tan^{-1}\left(\frac{r}{a}\right)\right) \;+\; \Phi_0$$

at a distance r from the centre, where G is the constant of gravitation, and ρ_0, a and Φ_0 are constants.

(a) What is the mass $M(r)$ interior to the radius r as a function of radius r?

(b) What is the circular velocity v_{circ} as a function of radius r?

What happens to v_{circ} when $r \gg a$? How does this compare with real galaxies?

(c) What is the density ρ as a function of radius r?

This profile has a particular name. What is the name of this profile?

(d) How would you interpret the constants ρ_0 and a?

(e) How does the potential $\Phi(r)$ behave as $r \to \infty$ and is this physically realistic?