Queen Mary University of London

Main Examination period 2023 – January – Semester A

MTH6106: Group Theory

Duration: 2 hours

The exam is intended to be completed within **2 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

All work should be **handwritten** and should **include your student number**. Only one attempt is allowed – **once you have submitted your work, it is final**.

In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;

Examiners: I.Morris, R. Johnson

Question 1 [25 marks].

(a) Give examples of groups with the following properties. (You do not need to prove that your examples have the properties claimed.)

(i) A group of order 24 which is not abelian. **[2]**

(ii) A group of infinite order which is not abelian. **[2]**

(iii) A pair of abelian groups of the same order which are not isomorphic to one another. **[2]**

(iv) A group G and two subgroups $H_1, H_2 \leq G$ such that $H_1 \cup H_2$ is not a subgroup of G. **[3]**

(b) Complete the following table in a way which results in the Cayley table of a group.

	1	a	b	c	d
1	1	a	b	c	d
a	a	b	c	d	1
b					
c					
d					

[5]

(c) The following table is **not** the Cayley table of a group. Indicate which group axioms are inconsistent with the operation defined by this table. For each group axiom which is inconsistent with the table, give an example of where in the table the inconsistency occurs.

	1	a	b	c	d
1	1	a	b	c	d
a	a	b	d	1	c
b	b	1	c	d	a
c	c	d	a	b	1
d	d	c	1	a	b

[5]

(d) Recall that $\mathrm{GL}_n(\mathbb{R})$ denotes the group of invertible $n \times n$ matrices with real entries. Let $\mathrm{O}(n)$ denote the set

$$\mathrm{O}(n) := \left\{ A \in \mathrm{GL}_n(\mathbb{R}) \colon A^T A = I \right\}$$

where I denotes the $n \times n$ identity matrix and A^T denotes the transpose of the matrix A. Show that $\mathrm{O}(n)$ is a subgroup of $\mathrm{GL}_n(\mathbb{R})$. **[6]**

Question 2 [25 marks].

(a) Let G be a group and let $f, g \in G$. Suppose that f and g have finite order and that $fg = gf$. Show that the order of fg is **less than or equal to** the least common multiple of the orders of f and g. [**3**]

(b) Give an example of two permutations $f, g \in S_3$ such that the order of fg is **not equal to** the least common multiple of the orders of f and g. [**3**]

(c) Consider the permutations $f, g \in S_8$ given by

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 5 & 6 & 4 & 3 & 1 & 8 & 7 & 2 \end{pmatrix}, \qquad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 8 & 1 & 4 & 7 & 5 & 2 & 6 \end{pmatrix}$$

Write f, g and fg in disjoint cycle notation and state the order of each of f, g and fg. [**9**]

(d) Let $n \geq 3$ and consider the group S_n.

(i) Show that every element of S_n can be written as a product of transpositions. [**2**]

(ii) Let $(1k), (1\ell) \in S_n$ be transpositions, where $2 \leq k, \ell \leq n$ and $k \neq \ell$. Write down the permutation $(1k)(1\ell)(1k)$ in disjoint cycle notation. [**2**]

(iii) Suppose that H is a subgroup of S_n which contains every transposition of the form $(1k)$, where $2 \leq k \leq n$. Explain why H must be equal to S_n. [**2**]

(iv) Suppose that H is a subgroup of S_n which contains the permutation (12) and also contains the permutation $(2345 \cdots n)$. Show that H contains every permutation of the form $(1k)$ where $2 \leq k \leq n$. [**3**]

(v) What is the group $\langle (12), (2345 \cdots n) \rangle$? [**1**]

Question 3 [25 marks].

(a) Suppose that G and H are finite groups and that $\phi \colon G \to H$ is a homomorphism.

(i) What information does Lagrange's theorem give you about the relationship between the numbers $|G|$, $|G/\ker\phi|$ and $|\ker\phi|$? [**2**]

(ii) What information does the First Isomorphism Theorem give you about the relationship between $|G/\ker\phi|$ and $|\operatorname{im}\phi|$? [**2**]

(iii) Indicate why $|\operatorname{im}\phi|$ divides both $|G|$ and $|H|$. [**3**]

(iv) Suppose that the numbers $|G|$ and $|H|$ are coprime. Prove that $\phi(g) = 1_H$ for all $g \in G$. [**2**]

(b) Let G be a finite group and let $H \leq G$. Show that for every $g \in G$, the set $gHg^{-1} = \{ghg^{-1} \colon h \in H\}$ is a subgroup of G. Now explain briefly why the following result holds: if H is the only subgroup of G with cardinality $|H|$, then H must be normal in G. [**5**]

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(c) Using Sylow's theorems, show that there is no simple group of order 51. **[5]**

(d) Using Sylow's theorems, or otherwise, show that A_5 has exactly 6 subgroups of order 5. State (without proof) any facts about A_5 which are required in your argument. **[6]**

Question 4 [25 marks].

(a) Consider the group $\mathrm{GL}_2(\mathbb{C})$ of all 2×2 complex matrices equipped with the usual operation of matrix multiplication. Recall that A^* denotes the conjugate transpose of the matrix A.

(i) Suppose that $\rho\colon\ \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_2(\mathbb{C})$ is an inner automorphism and let $A \in \mathrm{GL}_2(\mathbb{C})$. What is $\det(\rho(A))$? **[3]**

(ii) Show that the function $\hat{\rho}\colon\ \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_2(\mathbb{C})$ defined by $\hat{\rho}(A) = (A^*)^{-1}$ is an automorphism. **[4]**

(iii) Is the function $\hat{\rho}$ defined in (ii) an inner automorphism? Justify your answer with a proof or a counterexample. **[2]**

(b) Recall that $\mathcal{D}_{10} = \{1, r, \ldots, r^4, s, sr, \ldots, sr^4\}$ is the group of symmetries of a regular pentagon. Let X_5 denote the set of all possible colourings of the vertices of a regular pentagon using two colours, and let us say that two colourings are **equivalent** if one of them can be transformed into the other by applying a symmetry of the pentagon.

(i) How many elements does X_5 have? **[1]**

(ii) Suppose that $sr^k \in \mathcal{D}_{10}$ is a reflection, where $0 \leq k < 5$. How many elements of X_5 are stabilised by sr^k? **[3]**

(iii) Suppose that $r^k \in \mathcal{D}_{10}$ is a rotation, where $0 < k < 5$. How many elements of X_5 are stabilised by r^k? **[3]**

(iv) How many elements of X_5 are stabilised by $1 \in \mathcal{D}_{10}$? **[1]**

(v) Using the orbit-counting lemma, find the number of equivalence classes of elements of X_5. **[3]**

(vi) Now let $p \geq 3$ be an arbitrary prime number, let $\mathcal{D}_{2p}$ denote the group of symmetries of a regular p-sided polygon, and let X_p denote the set of all colourings of a regular p-sided polygon using two colours. We say that two elements of X_p are equivalent if one of them can be transformed into the other by the application of an element of $\mathcal{D}_{2p}$. By modifying the preceding argument, give a formula for the number of equivalence classes of elements of X_p and describe briefly how you obtained your formula. Indicate explicitly which step in your argument uses the fact that p is prime. **[5]**

End of Paper.