Queen Mary
University of London

Main Examination period 2020 – January – Semester A

MTH6106 / MTH6106P: Group Theory

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any unauthorised notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately.

It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it will be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiners: Matthew Fayers and Alex Fink

In this paper, we use the following notation.

- $\mathcal{C}_n$ denotes the cyclic group of order n.
- $\mathcal{U}_n$ is the set of integers between 0 and n which are prime to n, with the group operation being multiplication modulo n.
- $\mathcal{D}_{2n}$ is the group with $2n$ elements

$$1,\ r,\ r^2,\ \ldots,\ r^{n-1},\ s,\ rs,\ r^2s,\ \ldots,\ r^{n-1}s.$$

  The group operation is determined by the relations $r^n = s^2 = 1$ and $sr = r^{n-1}s$.
- $\mathcal{S}_n$ denotes the group of all permutations of $\{1, \ldots, n\}$, with the group operation being composition.
- $\mathrm{GL}_n(\mathbb{R})$ is the group of $n \times n$ invertible matrices with entries in $\mathbb{R}$, with the group operation being matrix multiplication.
- $\mathcal{Q}_8$ is the group $\{1, -1, i, -i, j, -j, k, -k\}$, in which

$$i^2 = j^2 = k^2 = -1, \qquad ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i,\ ik = -j.$$

Question 1 [21 marks].

(a) Give the definition of a **group**. **[3]**

(b) Give the definition of a **subgroup**. **[2]**

(c) Let

$$H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}) \;\middle|\; a + c = b + d \right\}.$$

Prove that H is a subgroup of $\mathrm{GL}_2(\mathbb{R})$. **[5]**

Suppose G is a group and $f, g \in G$.

(d) Prove that the inverse of g is unique. **[4]**

(e) Give the definition of the **order** of g. **[2]**

(f) Suppose g has order 4, and $gf = f^{-1}g$. What is the order of fg? *[Show your working.]* **[5]**

Question 2 [18 marks]. Suppose G is a group and $f, g \in G$.

(a) Define what it means to say that f and g are **conjugate** in G. **[2]**

(b) Give the definition of the **conjugacy class** of g in G. **[2]**

(c) Prove that if f and g are conjugate, then they have the same order. **[5]**

(d) Find all the elements in the conjugacy class of r^3 in $\mathcal{D}_{10}$. *[Show your working.]* **[5]**

(e) Write down five different elements of $\mathcal{S}_4$ of which no two are conjugate. *[You do not need to prove anything.]* **[4]**

Question 3 [12 marks]. Suppose G is a group, H is a subgroup of G and $g \in G$.

(a) Define what it means to say that H is **normal** in G. **[2]**

(b) Give the definition of the **right coset** Hg. **[2]**

(c) In the case where H is a normal subgroup of G, prove that $Hg = gH$. **[4]**

(d) Now suppose $G = \mathcal{U}_{21}$ and $H = \{1, 8, 13, 20\}$. Find all the right cosets of H in G. **[4]**

Question 4 [17 marks]. Suppose G and H are groups.

(a) Give the definition of a **homomorphism** from G to H. **[2]**

(b) Give the definition of an **automorphism** of G. **[2]**

(c) Give the definition of the **automorphism group** of G. **[2]**

(d) Find all the automorphisms of $\mathcal{C}_8$, and find the Cayley table for $\mathrm{Aut}(\mathcal{C}_8)$. *[Show your working.]* **[7]**

(e) Write down an automorphism of $\mathcal{Q}_8$ that maps i to $-j$. *[You do not have to prove anything, but you should say where each element of $\mathcal{Q}_8$ maps to.]* **[4]**

Question 5 [17 marks]. Suppose G is a group and X is a set.

(a) Give the definition of an **action** of G on X. **[3]**

(b) Give an example of a non-trivial action of $\mathcal{D}_8$ on itself which is not transitive. *[You do not need to prove anything, but you should make it clear how your action is defined.]* **[3]**

Suppose π is an action of G on X, and $x \in X$.

(c) Give the definition of the **orbit** of x. **[2]**

(d) Give a precise statement of the **Orbit-Counting Lemma**. **[3]**

(e) Suppose we colour the vertices and edges of an equilateral triangle, and we have n colours available. Say that two colourings are equivalent if one can be transformed into the other by applying a symmetry of the triangle. Use the Orbit-Counting Lemma to find the number of colourings up to equivalence. *[You should explain how you are using the Orbit-Counting Lemma as well as carrying out the calculation.]* **[6]**

Question 6 [15 marks]. Suppose G is a finite group and p is a prime number.

(a) Define what it means to say that G is **simple**. **[2]**

(b) Give the definition of a **Sylow p-subgroup of G.** **[2]**

(c) Find a Sylow 2-subgroup and a Sylow 3-subgroup of $\mathcal{U}_{11}$. **[4]**

(d) Give a precise statement of Sylow's Theorem 3 concerning the number of Sylow p-subgroups of a finite group. **[3]**

(e) Use this theorem to show that there is no simple group of order 44. **[4]**

End of Paper.