Queen Mary
University of London

Main Examination period 2019

MTH6104 / MTH6104P: Algebraic structures II

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Examiners: Matthew Fayers and Alex Fink

In this paper, we use the following notation.

- $\mathcal{C}_n$ denotes the cyclic group of order n.
- $\mathcal{U}_n$ is the set of integers between 0 and n which are prime to n, with the group operation being multiplication modulo n.
- $\mathcal{D}_{2n}$ is the group with $2n$ elements

$$1,\ r,\ r^2,\ \ldots,\ r^{n-1},\ s,\ rs,\ r^2s,\ \ldots,\ r^{n-1}s.$$

  The group operation is determined by the relations $r^n = s^2 = 1$ and $sr = r^{n-1}s$.
- $\mathcal{S}_n$ denotes the group of all permutations of $\{1, \ldots, n\}$ (with the group operation being composition).
- $\mathcal{Q}_8$ is the group $\{1, -1, i, -i, j, -j, k, -k\}$, in which

$$i^2 = j^2 = k^2 = -1, \qquad ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i,\ ik = -j.$$

Question 1 [20 marks].

(a) Give the definition of a **group**. **[3]**

Suppose G is a group and $f, g \in G$. In the rest of this question you may use elementary rules for manipulating powers of elements.

(b) Give the definition of the set $\langle g \rangle$, and prove that it is a subgroup of G. **[6]**

(c) In the case of the group $\mathcal{U}_{25}$, find all the elements of $\langle 6 \rangle$. **[4]**

(d) Give the definition of the **order** of g. **[2]**

(e) Suppose $\mathrm{ord}(f) = 3$, $\mathrm{ord}(g) = 4$ and $gf = f^2g$. What is the order of fg? Justify your answer. **[5]**

Question 2 [18 marks]. Suppose G is a group, $H, N \leqslant G$ and $g \in G$.

(a) Give the definition of the **right coset** Hg. **[2]**

(b) Find all the right cosets of the subgroup $\{1, 9, 31, 39\}$ in $\mathcal{U}_{40}$. **[4]**

(c) Define what it means to say that N is **normal** in G. **[2]**

(d) Now suppose N is a normal subgroup of G. Give the definition of the set NH, and prove that NH is a subgroup of G. **[6]**

(e) Give an example of a group G with $N, H \leqslant G$ such that NH is not a subgroup of G. *[You do not have to prove that N and H are subgroups, but you should show that NH is not a subgroup.]* **[4]**

Question 3 [13 marks].

(a) Give the definition of a **transposition** in $\mathcal{S}_n$. **[2]**

(b) Give the definition of the **alternating group** $\mathcal{A}_n$. **[2]**

(c) Suppose $h \in \mathcal{S}_n$. Explain how you can use the disjoint cycle notation for h to find the order of h and to find whether $h \in \mathcal{A}_n$. *[You do not need to prove anything.]* **[4]**

(d) Find an element of order 12 in $\mathcal{A}_9$, and write this element as a product of 3-cycles. *[You do not need to prove anything.]* **[5]**

Question 4 [20 marks]. Suppose G and H are groups.

(a) Give the definition of a **homomorphism** from G to H. **[2]**

(b) Does there exist a homomorphism $\phi : \mathcal{Q}_8 \to \mathcal{S}_4$ such that $\phi(i) = (1\,2\,3\,4)$ and $\phi(j) = (4\,3\,2\,1)$? Justify your answer. **[5]**

(c) Suppose $\phi : G \to H$ is a homomorphism. Give the definition of the **image** and **kernel** of ϕ. **[4]**

(d) Give a precise statement of the First Isomorphism Theorem. **[3]**

(e) Use the First Isomorphism Theorem to show that there is a normal subgroup K of $\mathcal{C}_{15}$ such that $\mathcal{C}_{15}/K \cong \mathcal{C}_5$. **[6]**

Question 5 [19 marks].

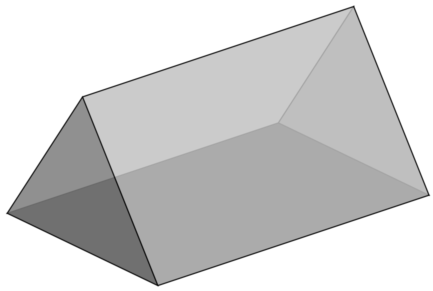
(a) Suppose G is a group and X is a set. Give the definition of an **action** of G on X. **[3]**

(b) Given an example of a transitive action of $\mathcal{Q}_8$ on $\mathcal{Q}_8$. *[You do not need to prove anything, but you should say clearly how the action is defined.]* **[3]**

(c) Suppose π is an action of G on X, and $x \in X$. Give the definition of the **orbit** containing x and the **stabiliser** of x. **[4]**

(d) Give a precise statement of the Orbit-Stabiliser Theorem. **[3]**

(e) Now let G be the symmetry group of a triangular prism (with the triangular faces being equilateral):



What is $|G|$? Justify your answer. **[6]**

Question 6 [10 marks]. Suppose G is a finite group and p is a prime number.

(a) Give the definition of a **Sylow p-subgroup** of G. **[2]**

(b) Find a Sylow 2-subgroup of $\mathcal{D}_{20}$. **[4]**

(c) Is this subgroup normal? Justify your answer. **[4]**

End of Paper.