Queen Mary University of London

MTH6104 / MTH6104P: Algebraic structures II

Duration: 2 hours

Date and time: 10th June 2016, 2:30pm

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You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best FOUR questions answered will be counted.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Examiner(s): Matthew Fayers

In this paper, we use the following notation.

- C_n denotes the cyclic group of order n.
- $\mathcal{U}_n$ is the set of integers between 0 and n which are prime to n, with the group operation being multiplication modulo n.
- $\mathcal{D}_{2n}$ is the group with $2n$ elements

$$1,\ r,\ r^2,\ \dots,\ r^{n-1},\ s,\ rs,\ r^2s,\ \dots,\ r^{n-1}s.$$

  The group operation is determined by the relations $r^n = s^2 = 1$ and $sr = r^{n-1}s$.

- Q_8 is the group $\{1, -1, i, -i, j, -j, k, -k\}$, in which

$$i^2 = j^2 = k^2 = -1, \qquad ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i,\ ik = -j.$$

In any question, you may freely use the Coset Lemma: if f, g are elements of a group G and $H \leqslant G$, then $Hf = Hg$ if and only if $fg^{-1} \in H$.

Question 1.

(a) Give the definition of a **group**. **[3]**

Suppose G is a group and $f, g \in G$.

(b) Give the definition of the **powers** g^n for $n \in \mathbb{Z}$. **[3]**

(c) Define what it means to say that f and g are **conjugate** in G. Give the definition of the **conjugacy class** containing g. **[4]**

(d) Is it true that if f is conjugate to g, then f^3 is conjugate to g^3? Justify your answer. **[5]**

(e) Is it true that if f^3 is conjugate to g^3, then f is conjugate to g? Justify your answer. **[5]**

(f) Find the conjugacy class containing s in $\mathcal{D}_{12}$. Show your working. **[5]**

Question 2. Write an essay on group actions. *[You should include precise definitions and statements of results, illustrated by examples and applications, and give some proofs.]* **[25]**

Question 3. Suppose G is a group, H is a subgroup of G and $g \in G$.

(a) Give the definition of the **right coset** Hg. **[2]**

(b) Give the definition of the **index** of H in G. **[2]**

(c) Suppose X is a right coset of H in G, and $a, b, c \in X$. Prove that $ab^{-1}c \in X$. **[4]**

(d) Give the definition of a **normal subgroup** of G. **[2]**

(e) Give an example of a group G and a subgroup H which is not normal. *[You do not need to prove anything.]* **[2]**

(f) Suppose N is a normal subgroup of G. Give the definition of the **quotient group** G/N. *[You do not have to prove that G/N is a group, but you should prove that the group operation is well-defined.]* **[6]**

(g) Find a subgroup N of $\mathcal{U}_{40}$ such that $|N| = 4$. Write down all the cosets of N in $\mathcal{U}_{40}$, and find the Cayley table of $\mathcal{U}_{40}/N$. *[You do not need to prove anything.]* **[7]**

Question 4. Suppose G and H are groups and $g \in G$.

(a) Give the definitions of the following terms:

- the **order** of g;
- a **homomorphism** from G to H;
- an **isomorphism** from G to H;
- an **automorphism** of G;
- an **inner automorphism** of G;
- the **automorphism group** of G. **[8]**

(b) Give an example of an automorphism of C_8 which is not the identity map. **[2]**

(c) Suppose $\phi : G \to H$ is an isomorphism, and $g \in G$. Prove that $\mathrm{ord}(\phi(g)) = \mathrm{ord}(g)$. *[You may assume that $\phi(1) = 1$ and that ϕ^{-1} is a homomorphism.]* **[4]**

(d) Write down a theorem describing the inner automorphism group of G in terms of the centre of G. *[You do not need to prove anything.]* **[3]**

Suppose ψ is an automorphism of Q_8 satisfying $\psi(-j) = i$ and $\psi(k) = j$.

(e) Write down $\psi(g)$ for every $g \in Q_8$, and hence find the order of ψ. **[5]**

(f) Is ψ an inner automorphism of Q_8? Justify your answer. **[3]**

Question 5.

(a) Give the definition of a **simple** group. *[You do not need to define what a group or a normal subgroup is.]* **[2]**

(b) Suppose G is an abelian group of order 49. Prove that G is not simple. *[You may use basic results about the orders of elements of G.]* **[6]**

(c) Now suppose G is a finite group and p is a prime. Give the definition of a **Sylow p-subgroup** of G. **[2]**

(d) Find a Sylow 2-subgroup, a Sylow 3-subgroup and a Sylow 5-subgroup of $\mathcal{D}_{12}$. *[You do not need to prove anything.]* **[5]**

(e) Give precise statements of all the Sylow Theorems. **[6]**

(f) Using the Sylow theorems, prove that there is no simple group of order 40. **[4]**

Question 6.

(a) Give the definition of the **symmetric group** $\mathcal{S}_n$ and of the **alternating group** $\mathcal{A}_n$. **[5]**

(b) Suppose $g \in \mathcal{S}_n$. Explain how to write g in **disjoint cycle notation**. **[3]**

(c) Explain how to use the disjoint cycle notation for g to find whether $g \in \mathcal{A}_n$. *[You do not need to prove anything.]* **[2]**

(d) Prove that any element of $\mathcal{A}_n$ can be written as a product of 3-cycles. **[5]**

(e) Now let G be any group, and suppose $H \leqslant G$ and $N \trianglelefteq G$. Define the set NH, and prove that it is a subgroup of G. **[5]**

(f) Now suppose $G = \mathcal{S}_4$ and $N = \{\mathrm{id}, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$. Write down a subgroup $H \leqslant \mathcal{S}_4$ different from N such that $|H| = 4$, and find all the elements of NH. *[You do not need to prove anything.]* **[5]**

End of Paper.